ARTICLE IN PRESS APAL:2569

Annals of Pure and Applied Logic ••• (••••) •••–•••

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Propositional team logics ☆

Fan Yang [a,*,1], Jouko Väänänen [b,c,2]

[a] Department of Values, Technology and Innovation, Delft University of Technology, Jaffalaan 5, 2628 BX Delft, The Netherlands
[b] Department of Mathematics and Statistics, Gustaf Hällströmin katu 2b, PL 68, FIN-00014 University of Helsinki, Finland
[c] University of Amsterdam, The Netherlands

ARTICLE INFO

Article history:
Received 21 June 2016
Received in revised form 14 December 2016
Accepted 11 January 2017
Available online xxxx

MSC:
03B60
03B70
03B05
68P15

Keywords:
Propositional team logics
Team semantics
Dependence logic
Non-classical logic

ABSTRACT

We consider team semantics for propositional logic, continuing [34]. In team semantics the truth of a propositional formula is considered in a *set* of valuations, called a *team*, rather than in an individual valuation. This offers the possibility to give meaning to concepts such as dependence, independence and inclusion. We associate with every formula ϕ based on finitely many propositional variables the set $[\![\phi]\!]$ of *teams* that satisfy ϕ. We define a maximal propositional team logic in which every set of teams is definable as $[\![\phi]\!]$ for suitable ϕ. This requires going beyond the logical operations of classical propositional logic. We exhibit a hierarchy of logics between the smallest, viz. classical propositional logic, and the maximal propositional team logic. We characterize these different logics in several ways: first syntactically by their logical operations, and then semantically by the kind of sets of teams they are capable of defining. In several important cases we are able to find complete axiomatizations for these logics.

1. Introduction

In classical propositional logic the propositional atoms, say $p_1, \ldots, p_n$, are given a truth value 1 or 0 by what is called a valuation and then any propositional formula ϕ can be associated with the set $|\phi|$ of valuations giving ϕ the value 1. This constitutes a perfect analysis of the circumstances under which ϕ is true. The formula ϕ can be presented in so-called Disjunctive Normal Form based on taking the disjunction of descriptions of the valuations in $|\phi|$. Two fundamental results can be proved for classical propositional

☆ Some results in this paper were included in the dissertation of the first author [32], which was supervised by the second author.

* Corresponding author.
E-mail addresses: fan.yang.c@gmail.com (F. Yang), jouko.vaananen@helsinki.fi (J. Väänänen).

1 This paper is partially based on research that was carried out in the Graduate School of Mathematics and Statistics of the University of Helsinki.
2 The research was partially supported by grant 251557 of the Academy of Finland.

http://dx.doi.org/10.1016/j.apal.2017.01.007

logic. The first says that *every* set of valuations of $p_1, \ldots, p_n$ is equal to $|\phi|$ for some propositional formula ϕ. The second fundamental result says that there is a simple *complete* axiomatization of those ϕ that are valid in the sense that $|\phi|$ is the full set of all valuations on the propositional atoms occurring in ϕ.

In this paper, which continues [34], we consider a richer semantics called *team semantics* for propositional logic. In team semantics the truth of a propositional formula is evaluated in a *set* of valuations, called a *team*, rather than in an individual valuation. This offers the possibility of considering *probabilities* of formulas, as in [18], and the meaning of concepts such as *dependence, independence* and *inclusion*, as in [34]. It is the latter possibility that is our focus in this paper.

Team semantics was introduced by the second author in [27] on the basis of a new compositional semantics, due to Hodges [16,17], for independence friendly logic [15,24]. The monograph [27] was written in the context of predicate logic and team semantics was used to give meaning to a variable being totally determined by a sequence of other variables. In the context of propositional and modal logic team semantics was introduced in [28]. In propositional logic team semantics can be used to give meaning to a propositional variable being totally determined by a sequence of other variables. It took a few years before this idea was fully exploited in [31,32]. Meanwhile modal dependence logic, i.e. team semantics for modal logic, was investigated e.g. in [6–8,13,14,22,25].

When propositional formulas are evaluated in a team—i.e. a set—of valuations, a whole new landscape opens in front of us. The first observation is a numerical explosion: If we have n propositional atoms, there are 2^n valuations, 2^{2^n} teams, and $2^{2^{2^n}}$ sets of teams. For $n = 3$ the third number is about 10^{77}. This emphasizes the need for mathematical methods in team semantics. The truth table methods which list all possibilities is bad enough in ordinary propositional logic, but totally untenable in team semantics.

In classical propositional logic, we associate with every formula ϕ based on propositional atoms $p_1, \ldots, p_n$ the set $|\phi|$ of *valuations* that satisfy ϕ. Similarly, in team semantics we associate with every formula ϕ based on propositional atoms $p_1, \ldots, p_n$ the set $[\![\phi]\!]$ of *teams* that satisfy (in the sense defined below) ϕ. By choosing our formulas carefully we can express *every* set of teams in the form $[\![\phi]\!]$ for suitable ϕ, but this requires going beyond the logical operations of classical propositional logic. We can also axiomatize the propositional formulas that are *valid* i.e. satisfied by every team.

The rich structure of teams gives rise to a plethora of new propositional connectives. Most importantly, disjunction has several versions. To define when a team X satisfies $\phi \vee \psi$ we can say that this happens if X satisfies ϕ or it satisfies ψ, or we can say that this happens if X is the union of two sets Y and Z such that Y satisfies ϕ and Z satisfies ψ, or, finally, we can also say that this happens if, assuming $X \neq \emptyset$, the team X is the union of two sets $Y \neq \emptyset$ and $Z \neq \emptyset$ such that Y satisfies ϕ and Z satisfies ψ. If X is a singleton, which corresponds to the classical case, the first two disjunctions are equivalent, but the third is equivalent to $\phi \wedge \psi$. But for non-singleton teams there is a big difference in every respect. These distinctions, leading to different variants of familiar logical operations, reveal a hierarchy of logics between the smallest, viz. classical propositional logic, and the maximal one capable of defining every set of teams. We characterize these different logics in several ways: first syntactically by their logical operations, and then semantically by the kind of sets of teams they are capable of defining. In several important cases we are able to find complete axiomatizations for these logic.

In our previous paper [34] we considered sets of teams that are *downward closed* in the sense that if a team is in the set, then every subteam is in the set, too. Respectively, the logics studied in [34] have the property that the sets of teams defined by their formulas are downward closed. We isolated five equivalent logics with this property, all based on some aspect of *dependence*. In these logics every downward closed set of teams is definable, and the logics have complete axiomatizations. The axiomatizations are by no means as simple as typical axiomatizations of classical propositional logic, but have still a certain degree of naturality.

In this paper we consider sets of teams, and related propositional logics, that are not downward closed. A property in a sense opposite to downward closure is closure under (set-theoretical) unions. In fact, a set of teams that is both closed downward and closed under unions is definable in classical propositional logic.