Annals of Pure and Applied Logic 168 (2017) 1442–1471

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Neostability in countable homogeneous metric spaces

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ARTICLE INFO

Article history:
Received 25 October 2015
Received in revised form 31 March 2016
Accepted 22 January 2017

MSC:
03C45
03C15
54E35
06F05

Keywords:
Generalized metric space
Neostability
Strong order property

ABSTRACT

Given a countable, totally ordered commutative monoid $\mathcal{R} = (R, \oplus, \leq, 0)$, with least element 0, there is a countable, universal and ultrahomogeneous metric space $\mathcal{U}_\mathcal{R}$ with distances in $\mathcal{R}$. We refer to this space as the $\mathcal{R}$-*Urysohn space*, and consider the theory of $\mathcal{U}_\mathcal{R}$ in a binary relational language of distance inequalities. This setting encompasses many classical structures of varying model theoretic complexity, including the rational Urysohn space, the free nth roots of the complete graph (e.g. the random graph when $n = 2$), and theories of refining equivalence relations (viewed as ultrametric spaces). We characterize model theoretic properties of $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ by algebraic properties of $\mathcal{R}$, many of which are first-order in the language of ordered monoids. This includes stability, simplicity, and Shelah's SOP_n-hierarchy. Using the submonoid of idempotents in $\mathcal{R}$, we also characterize superstability, supersimplicity, and weak elimination of imaginaries. Finally, we give necessary conditions for elimination of hyperimaginaries, which further develops previous work of Casanovas and Wagner.

1. Introduction

In this paper, we consider model theoretic properties of generalized metric spaces obtained as analogs of the rational Urysohn space. Our results will show that this class of metric spaces exhibits a rich spectrum of complexity in the classification of first-order theories without the strict order property.

The object of focus is the countable $\mathcal{R}$-*Urysohn space*, denoted $\mathcal{U}_\mathcal{R}$, where $\mathcal{R} = (R, \oplus, \leq, 0)$ is a countable totally ordered commutative monoid with least element 0, or *distance monoid* (see Definition 2.5). We refer to generalized metric spaces taking distances in $\mathcal{R}$ as $\mathcal{R}$-*metric spaces*. The space $\mathcal{U}_\mathcal{R}$ is then defined to be the unique countable, ultrahomogeneous $\mathcal{R}$-metric space, which is universal for finite $\mathcal{R}$-metric spaces. Explicitly, $\mathcal{U}_\mathcal{R}$ is the Fraïssé limit of the class of finite $\mathcal{R}$-metric spaces, and its existence follows from the work in the prequel [11] to this paper, which generalizes previous results of Delhommé, Laflamme, Pouzet, and Sauer [13]. In particular, associativity of $\oplus$ is not required to define $\mathcal{R}$-metric spaces and, in

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http://dx.doi.org/10.1016/j.apal.2017.01.008

fact, characterizes when the class of finite $\mathcal{R}$-metric spaces is a Fraïssé class (see [25, Theorem 5] or [11, Proposition 5.7]). We give a few examples of historical and mathematical significance.

Example 1.1.

1. Let $\mathcal{Q} = (\mathbb{Q}^{\geq 0}, +, \leq, 0)$ and $\mathcal{Q}_1 = (\mathbb{Q} \cap [0, 1], +_1, \leq, 0)$, where $+_1$ is addition truncated at 1. Then $\mathcal{U}_{\mathcal{Q}}$ and $\mathcal{U}_{\mathcal{Q}_1}$ are, respectively, the *rational Urysohn space* and *rational Ursyohn sphere*. The completion of $\mathcal{U}_{\mathcal{Q}}$ is called the *Urysohn space*, and is the unique complete, separable metric space, which is homogeneous and universal for separable metric spaces. The completion of $\mathcal{U}_{\mathcal{Q}_1}$ is called the *Urysohn sphere* and satisfies the same properties with respect to separable metric spaces of diameter ≤ 1. These spaces were originally constructed by Urysohn in 1925 (see [29,30]).
2. Let $\mathcal{R}_2 = (\{0, 1, 2\}, +_2, \leq, 0)$, where $+_2$ is addition truncated at 2. Then $\mathcal{U}_{\mathcal{R}_2}$ is isometric to the *countable random graph* or *Rado graph* (when equipped with the path metric). A directed version of this graph was first constructed by Ackermann in 1937 [1]. The undirected construction is usually attributed to Erdős and Rényi [15, 1963] or Rado [24, 1964].
3. Generalize the previous example as follows. Fix $n > 0$ and let $\mathcal{R}_n = (\{0, 1, \dots, n\}, +_n, \leq, 0)$, where $+_n$ is addition truncated at n. Let $\mathcal{N} = (\mathbb{N}, +, \leq, 0)$. We refer to $\mathcal{U}_{\mathcal{R}_n}$ as the *integral Urysohn space of diameter* n, and to $\mathcal{U}_{\mathcal{N}}$ as the *integral Urysohn space*. These spaces were constructed by Pouzet and Roux [23, 1996] and Cameron [5, 1998]. Also, Casanovas and Wagner [7, 2004] construct the *free nth root of the complete graph*. As with $n = 2$, equipping this graph with the path metric yields $\mathcal{U}_{\mathcal{R}_n}$.
4. Generalize all of the previous examples as follows. Fix a countable subset $S \subseteq \mathbb{R}^{\geq 0}$ closed under the operation $r +_S s := \sup\{x \in S : x \leq r + s\}$ and containing 0. Let $\mathcal{S} = (S, +_S, \leq, 0)$ and assume $+_S$ associative. Then we have the $\mathcal{S}$-Urysohn space $\mathcal{U}_{\mathcal{S}}$. This situation is studied in further generality by Delhommé, Laflamme, Pouzet, and Sauer [13].
5. For an example of a different flavor, fix a countable linear order $(R, \leq, 0)$, with least element 0, and let $\mathcal{R} = (R, \max, \leq, 0)$. We refer to $\mathcal{U}_{\mathcal{R}}$ as the *ultrametric Urysohn space over* $(R, \leq, 0)$. Explicit constructions of these spaces are given by Gao and Shao in [16]. Alternatively, $\mathcal{U}_{\mathcal{R}}$ is a countable model of the theory of infinitely refining equivalence relations indexed by $(R, \leq)$. These are standard model theoretic examples, often used to illustrate various behavior in the stability spectrum (see [4, Section III.4]).

We will consider model theoretic properties of $\mathcal{R}$-Urysohn spaces. In particular, given a countable distance monoid $\mathcal{R}$, we let $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ be the complete $\mathcal{L}_R$-theory of $\mathcal{U}_{\mathcal{R}}$, where $\mathcal{L}_R$ is a first-order language consisting of binary relations $d(x, y) \leq r$, for $r \in R$. In [11], we constructed a "nonstandard" distance monoid extension $\mathcal{R}^*$ of $\mathcal{R}$, with the property that any model of $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ is canonically an $\mathcal{R}^*$-metric space (see Theorem 2.7 below). We let $\mathbb{U}_{\mathcal{R}}$ denote a sufficiently saturated monster model of $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$. Then, as an $\mathcal{R}^*$-metric space, $\mathbb{U}_{\mathcal{R}}$ is κ^+-universal, where κ is the saturation cardinal (see [11, Proposition 6.1]). However, in order to conclude that $\mathbb{U}_{\mathcal{R}}$ is also κ-homogeneous as an $\mathcal{R}^*$-metric space, we must assume quantifier elimination. Therefore we say that $\mathcal{R}$ is a **Urysohn monoid** if it is a countable distance monoid and $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ has quantifier elimination. In [11, Theorem 6.10], we characterized quantifier elimination via continuity of addition in $\mathcal{R}^*$ (see Theorem 2.8 below). This motivates a general schematic for analyzing the model theoretic behavior of $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$.

Definition 1.2. Let **RUS** denote the class of $\mathcal{R}$-Urysohn spaces $\mathcal{U}_{\mathcal{R}}$, where $\mathcal{R}$ is a Urysohn monoid. We say a property P of **RUS** is **axiomatizable** (resp. **finitely axiomatizable**) if there is an $\mathcal{L}_{\omega_1,\omega}$-sentence (resp. $\mathcal{L}_{\omega,\omega}$-sentence) φ_P, in the language of ordered monoids, such that, if $\mathcal{R}$ is a Urysohn monoid, then $\mathcal{U}_{\mathcal{R}}$ satisfies P if and only if $\mathcal{R} \models \varphi_P$.