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A uniform version of non-low$_2$-ness

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ABSTRACT

We introduce a property of Turing degrees: being uniformly non-low$_2$. We prove that, in the c.e. Turing degrees, there is an incomplete uniformly non-low$_2$ degree, and not every non-low$_2$ degree is uniformly non-low$_2$. We also build some connection between (uniform) non-low$_2$-ness and computable Lipschitz reducibility ($\leq_{cl}$), as a strengthening of weak truth table reducibility:
(1) If a c.e. Turing degree **d** is uniformly non-low$_2$, then for any non-computable Δ_2^0 real there is a c.e. real in **d** such that both of them have no common upper bound in c.e. reals under cl-reducibility.
(2) A c.e. Turing degree **d** is non-low$_2$ if and only if for any Δ_2^0 real there is a real in **d** which is not cl-reducible to it.

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1. Introduction

Our paper starts with recalling array computable Turing degrees and computable Lipschitz reducibility. Following [5], a degree **d** is array computable if there is a total function $f \leq_{wtt} \emptyset'$ which dominates all total functions computable in **d**. The class of array computable degrees has caused significant recent interest. It has deep connections with algorithmic randomness, as witnessed by, for example, Kummer [8], Downey and Greenberg [3], Barmpalias, Downey and Greenberg [2], and several other studies (see Downey and Hirschfeldt [4]).

Computable Lipschitz reducibility [6,9,4] is a strengthening of weak truth table reducibility, based on computations where the use on the oracle at argument n is bounded by $n + c$ for some constant c. We adopt this terminology in [9,4] and note it as $\leq_{cl}$. Schnorr's theorem proved that A is Martin-Löf random if and only if for all n, $K(A \upharpoonright\upharpoonright n) = n + \mathrm{O}(1)$, where $A \upharpoonright\upharpoonright n$ denotes the first n bits of A and K denotes prefix free Kolmogorov complexity. Schnorr's theorem suggests a natural method of calibrating randomness of reals: $A \leq_K B$ iff for all n, $K(A \upharpoonright\upharpoonright n) \leq K(B \upharpoonright\upharpoonright n) + \mathrm{O}(1)$. It is obvious that $\leq_{cl}$ is a measure of relative

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randomness since a program computing $n + c$ bits of B will compute n bits of A. For a more complete account of the role by cl-reducibility in the theory of algorithmic randomness, we refer the reader to the monograph by Downey and Hirschfeldt [4].

In [13], Yu and Ding proved that there are two c.e. reals (i.e. limits of computable increasing sequences of rationals) which have no common upper bound in the c.e. reals under cl-reducibility. In [1], Barmpalias and Lewis showed that there is a c.e. real which is not cl-reducible to any Martin-Löf random c.e. real. In the c.e. Turing degrees both the Yu-Ding theorem and the Barmpalias–Lewis theorem can characterize array computability as follows [2].

Theorem 1.1 *(Barmpalias, Downey and Greenberg [2]). The following are equivalent for a c.e. Turing degree* $\mathbf{d}$*,*

(1) *There are c.e. reals* $\alpha, \beta \in \mathbf{d}$ *which have no common upper bound in the cl-degrees of c.e. reals.*
(2) *There is a c.e. real* $\beta \in \mathbf{d}$ *which is not cl-reducible to any random c.e. real.*
(3) *There is a set* $A \in \mathbf{d}$ *which is not cl-reducible to any random c.e. real.*
(4) $\mathbf{d}$ *is array non-computable.*

Theorem 1.1 leads us to the theme: the interplay between Turing and cl reducibility. Recall that a degree $\mathbf{d}$ is non-low$_2$ if for any total function $f \leq_T \emptyset'$ there is a total function computable in $\mathbf{d}$ which is not dominated by f. It would be quite interesting to gain a better understanding of non-low$_2$-ness via cl-reducibility. In [7], Fan and Yu proved that for any non-computable Δ^0_2 real α there is a c.e. real β such that α and β have no common upper bound in the c.e. reals under cl-reducibility. Is there such a c.e. real β in each non-low$_2$ degree? To answer this question, we introduce a new class of Turing degrees: a uniform version of non-low$_2$-ness.

Definition 1.2. A Turing degree $\mathbf{d} = deg_T(D)$ is *uniformly non-low*$_2$ if for any total function $f \leq_T \emptyset'$ there is a uniform way to define a total function $g \leq_T D$ such that g is not dominated by f. In more detail, there is a computable function p such that given an index e, if $f = \Phi_e^{\emptyset'}$ is total then $g = \Phi^D_{p(e)}$ is total and not dominated by f.

In this paper, we focus on the c.e. Turing degrees. From now on, each degree mentioned is a c.e. Turing degree.

Firstly, we prove that this uniform non-low$_2$-ness is not trivial as follows.

Proposition 1.3. *There is an incomplete uniformly non-low*$_2$ *c.e. degree* $\mathbf{d}$*.*

Secondly, each uniformly non-low$_2$ degree is non-low$_2$, but not conversely.

Proposition 1.4. *There is a non-low*$_2$ *c.e. degree* $\mathbf{d}$*, which is not uniformly non-low*$_2$*.*

We connect uniform non-low$_2$-ness and computable Lipschitz reducibility as follows.

Theorem 1.5. *If a c.e. degree* $\mathbf{d}$ *is uniformly non-low*$_2$*, then for any non-computable* Δ^0_2 *real* α *there is a c.e. real* $\beta \in \mathbf{d}$ *such that* α *and* β *have no common upper bound in c.e. reals under cl-reducibility. Furthermore, an index for* β *can be found uniformly from one of* α*.*

Obviously, in Theorem 1.5 the real $\beta \in \mathbf{d}$ is not cl-reducible to α, which assures that α is not cl-complete in the c.e. reals. Recall the version of the Yu-Ding Theorem in the Δ^0_2 reals is proven in [10]: there are two

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