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An ordered credibility contrast semantics for finite probability agreement

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ABSTRACT

De Finetti's 1949 ordinal probability conjecture sparked enduring interest in intuitively meaningful necessary and sufficient conditions for orderings of finite propositional domains to agree with probability distributions. This paper motivates probabilistic ordering from subjective estimates of credibility contrasts revealed when ordered propositions are not monotonically related (e.g., A or $B > C$ or D, but $D > B$) and when a portfolio of prospects is accepted as preferable to alternatives despite not dominating them. The estimated contrast primitive offers a gambling-free, psychologically grounded foundation for treating individual instances and multisets of propositions as credally interchangeable with disjunctions and multisets of their constituent atomic propositions.

1. Introduction

In 1949, George Polya [11] argued for qualitative, non-numerical and not necessarily probabilistic belief representations for general-purpose uncertainty management. In the same issue of the journal *Dialectica*, Bruno de Finetti [4] gave a summary of his own domain-independent, thoroughly numerical gambling semantics for subjective probability. Although not directed at any specific person's work, Polya's paper had the quality of a rebuttal to de Finetti.

De Finetti responded to Polya later in the year with a conference paper [5] in which he conjectured that any ordering of the propositions in a finite domain which was quasi-additive would be probability agreeing. If so, then quasi-additive orderings could provide an intuitively appealing foundation for qualitative uncertain reasoning.

Definition and conventions. A *propositional domain* comprises the empty disjunction, denoted by ∅, and all disjunctions compounded from one or more selections from a set of exclusive and exhaustive propositions called "atoms." This paper discusses only domains compounded from a finite set of atoms whose propositions

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are completely, definitely and transitively ordered. That is, for any propositions A, B and C, exactly one of $A > B$, $A < B$ or $A = B$ obtains, and if $A \geq B$ and $B \geq C$, then $A \geq C$, where if either antecedent inequality is strict, then so is the consequent.

A propositional ordering is *quasi-additive* just when for all propositions A, B and C where $AB = AC = \varnothing$, $A \vee B \geq A \vee C$ just when $B \geq C$. A propositional domain is *probability agreeing* just when there is a probability distribution $p()$ over the propositions such that for all propositions A, B, $A \geq B$ just when $p(A) \geq p(B)$.

There is also interest in partially ordered domains of propositions, or indefinitely ordered domains. However, this paper emphasizes the ordering of collections and combinations of propositions. Completeness and definiteness of the underlying propositional ordering avoids unilluminating complications and allows focus on the properties of the collections and combinations.

Kraft, Pratt and Seidenberg [9] showed by counterexample that de Finetti's qualitative conjecture was false.

Example. The following ordering assertions are taken from a quasi-additive complete ordering of the thirty-two distinct propositions built by disjunction of zero or more of the five atoms $\{a, b, c, d, e\}$.

$$a \vee e > c \vee d; \quad a \vee c \vee d > b \vee e; \quad b \vee c > a \vee d; \quad d > a \vee c$$

The ordering is not probability agreeing. If there were some $p()$ that agreed with the above, then

$$p(a) + p(e) > p(c) + p(d) \quad \text{and} \quad p(a) + p(c) + p(d) > p(b) + p(e)$$
$$\text{and} \quad p(b) + p(c) > p(a) + p(d) \quad \text{and} \quad p(d) > p(a) + p(c)$$

which sum to the single inequality $2p(a) + p(b) + 2p(c) + 2p(d) + p(e) >$ itself, which is a contradiction.

The refutation of de Finetti's conjecture did not extinguish interest in quasi-additive orderings on finite domains as an object of study in their own rights, whether or not they are probability agreeing [3,8,10]. However, this paper focuses on finite probability agreement, that is, de Finetti's goal, rather than the means by which he had tried to achieve it.

Kraft, et al. also proved a version of the usual "technical" condition for probability agreement, which Scott [12] later derived in the context of de Finetti's conjecture using a different method. The condition is also directly verifiable by ordinary principles governing the consistency of simultaneous linear inequality systems, of which probabilistic ordering assertions are a simple special case.

Theorem 1 *(Kraft, Pratt and Seidenberg; Scott; and standard result). An ordering of the propositions in a finite domain $\{A, B, C, D, E, F, \ldots\}$ is probability-agreeing just when for every finite ensemble of ordering assertions where at least one assertion is strict,*

$$A > B, \quad C \geq D, \quad E \geq F, \quad etc.$$

at least one atom in the domain appears more often among the favored propositions (A, C, E, etc.) than among the disfavored propositions (B, D, F, etc.).

At first glance, there may be nothing intuitively appealing about comparing the number of times each atom appears on both sides of the inequalities. As William Kingdon Clifford [2] famously described the resulting empty feeling, "We may always depend on it that algebra, which cannot be translated into good English and sound common sense, is bad algebra."

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