Journal of Applied Logic ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Applied Logic

www.elsevier.com/locate/jal

# Characterization of unidimensional averaged similarities

D. Boixader *, J. Recasens

*Secció de Matemàtiques i Informàtica, Dept. Tecnologia de l'Arquitectura, Universitat Politècnica de Catalunya, Spain*

ARTICLE INFO

*Article history:*
Available online xxxx

*Keywords:*
Indistinguishability operator
Generator
Quasi-arithmetic mean
Representation theorem

ABSTRACT

A $T$-indistinguishability operator (or fuzzy similarity relation) $E$ is called unidimensional when it may be obtained from one single fuzzy subset (or fuzzy criterion). In this paper, we study when a $T$-indistinguishability operator that has been obtained as an average of many unidimensional ones is unidimensional too. In this case, the single fuzzy subset used to generate $E$ is explicitly obtained as the quasi-arithmetic mean of all the fuzzy criteria primarily involved in the construction of $E$.


## 1. Introduction

Indistinguishability operators with respect to a given $t$-norm $T$, or simply $T$-indistinguishability operators, are the natural fuzzification of classical equivalence relations. They are found under many names in the literature, depending on the author and on the chosen $t$-norm. *Similarity* is perhaps the most common name applied to such fuzzy relations (Zadeh [7]), although it is sometimes associated with the particular $t$-norm $T = MIN$. Other names are *Likeness*, *Fuzzy Equality* or *Fuzzy Equivalence Relation*. We will use *$T$-indistinguishability operator* (following Trillas and Valverde [6]), and also the term *similarity* in an informal way.

Crisp equivalence relations are generally regarded as the mathematical construct for dealing with classifications. They are defined as those relations being reflexive, symmetric and transitive. If $E$ is such a relation on a set $X$, for each element $x \in X$ we may consider all the elements $y \in Y$ that are related to $x$, that is, all $y \in Y$ such that $E(x, y) = 1$. These elements are *the class of* $x$. Here $x$ acts as a prototype, and all the objects $y$ in its class as its likes. As a result, the set $X$ becomes partitioned into classes.

Often, equivalence relations are induced by attributes. For example, a given set $X$ of plane closed polygonal lines becomes naturally partitioned into classes according to their number of sides. In addition to that, if the polygonal are real (drawn) lines, we may consider also color as an attribute, and the set becomes furtherly partitioned into, say, black and white lines. Each attribute is responsible for a partition of $X$ and,

* Corresponding author.
*E-mail address:* dionis.boixader@upc.edu (D. Boixader).

http://dx.doi.org/10.1016/j.jal.2016.11.003
1570-8683/

therefore, for an equivalence relation $E$. The final partition or equivalence relation is the intersection of the two, meaning with this that every two elements $x$ and $y \in X$ are $E$-related if they have the same number of sides and the same color, but they are not if they fail to meet one of the two criteria, or both of them. Formally, if $E_s$ stands for *number of sides* and $E_c$ means *color* then $E(x,y) = MIN\,(E_s(x,y), E_c(x,y))$.

Attributes, though, may be of a graded nature. We may consider the attribute *perimeter* of a polygonal, whose range is the positive real numbers, or lines may be drawn in a variety of shades of gray which can be expressed as real numbers between 0 and 1. Attributes that take values on continuous universes are generally regarded as vague, and they are represented by fuzzy sets. Instead of considering a rectangle whose perimeter equals 5 as entirely different from another one of perimeter 5.15, and therefore belonging to two different classes, we can regard them as very similar objects whenever perimeter is the only attribute considered. They could share the same class, provided that classes are fuzzy sets and belonging to a class is a matter of degree.

The definition of $T$-indistinguishability operator axiomatically captures the intuitive idea of fuzzy equivalence relation.

**Definition 1.1.** Let $X$ be a universe and $T$ a t-norm. A $T$-indistinguishability operator $E$ on $X$ is a fuzzy relation $E : X \times X \to [0,1]$ satisfying, for all $x, y, z \in X$,

1. $E(x,x) = 1$ (Reflexivity);
2. $E(x,y) = E(y,x)$ (Symmetry);
3. $T(E(x,y), E(y,z)) \leq E(x,z)$ ($T$-Transitivity).

A *t-norm* $T$ is an operation on the unit interval which is associative, commutative, nondecreasing in both variables, and satisfies the boundary condition $T(x,1) = x$ for all $x, y \in [0,1]$. It is generally accepted that $t$-norms are the *AND* connectives of Fuzzy Logic [2].

We will assume within this paper that the $t$-norm $T$ is continuous and Archimedean [3], or else $T = MIN$. Every continuous Archimedean t-norm is isomorphic to the sum of positive real numbers, bounded or unbounded, according to Ling's theorem [3]. The order reversing isomorphism $t : [0,1] \to [0,+\infty]$ is called *an additive generator of* $T$, and $T(a,b) = t^{[-1]}(t(a)+t(b))$ for all $a, b \in [0,1]$ where $t^{[-1]}$ is the pseudoinverse of $t$.

In practice, this means that $T$-transitivity (Definition 1.1.3) is simply a version of the *triangle inequality* for metrics, since $T\left(E\left(x,y\right), E\left(y,z\right)\right) \leq E\left(x,z\right)$ can be rewritten as $t\left(E\left(x,y\right)\right) + t\left(E\left(y,z\right)\right) \geq t\left(E\left(x,z\right)\right)$ or, in a more convenient notation for the purposes of this paper,

$$t \circ E\left(x,y\right) + t \circ E\left(y,z\right) \geq t \circ E\left(x,z\right).$$

Thus, the underlying semantics of $T$-indistinguishability operators is enhanced to include proximity in a metric sense in addition to fuzzy equivalence.

$T$-indistinguishability operators may be induced by fuzzy attributes. These fuzzy attributes may be represented as fuzzy sets $h : X \to [0,1]$, and then some procedure is needed to obtain the fuzzy relation $E$ from the fuzzy subsets $h$. Such procedure is provided by the Representation Theorem [6]

For a given continuous $t$-norm $T$, we consider the *natural indistinguishability* on $[0,1]$, $E_T$. Since in this paper only the most used $t$-norms, namely Archimedean $t$-norms and the *MIN* t-norm are considered, we will provide separate definitions for each of the two, thus avoiding the general case [6].

**Definition 1.2.** Given an Archimedean $t$-norm $T$, the natural indistinguishability $E_T$ associated with $T$ is the indistinguishability on the unit interval $E_T\left(x,y\right) = t^{[-1]}\left(\left|t\left(x\right) - t\left(y\right)\right|\right)$.