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# Coloring trees in reverse mathematics ☆

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ABSTRACT

The tree theorem for pairs ($\mathsf{TT}^2_2$), first introduced by Chubb, Hirst, and McNicholl, asserts that given a finite coloring of pairs of comparable nodes in the full binary tree $2^{<\omega}$, there is a set of nodes isomorphic to $2^{<\omega}$ which is homogeneous for the coloring. This is a generalization of the more familiar Ramsey's theorem for pairs ($\mathsf{RT}^2_2$), which has been studied extensively in computability theory and reverse mathematics. We answer a longstanding open question about the strength of $\mathsf{TT}^2_2$, by showing that this principle does not imply the arithmetic comprehension axiom ($\mathsf{ACA}_0$) over the base system, recursive comprehension axiom ($\mathsf{RCA}_0$), of second-order arithmetic. Combined with a recent result of Patey's that $\mathsf{TT}^2_2$ is strictly stronger than $\mathsf{RT}^2_2$, this establishes $\mathsf{TT}^2_2$ as the first known example of a natural combinatorial principle to occupy the interval strictly between $\mathsf{ACA}_0$ and $\mathsf{RT}^2_2$. The proof of this fact uses an extension of the bushy tree forcing method, and develops new techniques for dealing with combinatorial statements formulated on trees, rather than on $\omega$.



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## 1. Introduction

Reverse mathematics is an area of mathematical logic devoted to classifying mathematical theorems according to their logical strength. The setting for this endeavor is *second-order arithmetic* which is a formal system strong enough to encompass (countable analogues of) most results of classical mathematics. It consists of the usual Peano axioms for the natural numbers, together with the *comprehension scheme*, consisting of axioms asserting that the set of all $x \in \mathbb{N}$ satisfying a given formula $\varphi$ exists. Fragments of this system obtained by weakening the comprehension scheme are called *subsystems of second-order arithmetic*. The logical strength of a theorem is then measured according to the weakest such subsystem in which that theorem can be proved. This is a two-step process: the first consists in actually finding such a subsystem, and the second in showing that the theorem "reverses", i.e., is in fact equivalent to this subsystem, over a fixed weak base system. One way to think about such a reversal is that it precisely captures the techniques needed to prove the given theorem. By extension, two theorems that turn out to be equivalent to the same subsystem (and hence to each other) can thus be regarded as requiring the same basic techniques to prove. The observation mentioned above, that most theorems can be classified into just a few categories, refers to the fact that most theorems are either provable in the weak base system, or are equivalent over it to one of four other subsystems.

The base system here is the *recursive comprehension axiom* ($\mathsf{RCA}_0$), which restricts the comprehension scheme to $\Delta^0_1$-definable sets. This system corresponds roughly to constructive mathematics, sufficing to prove the existence of all the computable sets, but not of any noncomputable ones. A considerably stronger system is the *arithmetical comprehension axiom* ($\mathsf{ACA}_0$), which adds comprehension for sets definable by arithmetical formulas, i.e., formulas whose quantifiers only range over variables for numbers (as opposed to variables for sets of numbers). This system suffices to solve the halting problem, i.e., the problem of determining whether a given computer program halts on a given input, and as such is considerably stronger than $\mathsf{RCA}_0$. Three other important systems are *weak König's lemma* ($\mathsf{WKL}_0$), *arithmetical transfinite recursion* ($\mathsf{ATR}_0$), and the $\Pi^1_1$-*comprehension axiom* ($\Pi^1_1$-$\mathsf{CA}_0$). In order of increasing strength, these are arranged thus:

$$\mathsf{RCA}_0 < \mathsf{WKL}_0 < \mathsf{ACA}_0 < \mathsf{ATR}_0 < \Pi^1_1\text{-}\mathsf{CA}_0.$$

We refer the reader to Simpson [33] for a complete treatise on reverse mathematics, and to Soare [34] for general background on computability theory.

A striking observation, repeatedly demonstrated in the literature, is that most theorems investigated in this framework are either provable in the base system $\mathsf{RCA}_0$, or else equivalent over $\mathsf{RCA}_0$ to one of the other four subsystems listed above. It is from this empirical fact that these systems derive their commonly-used moniker, "the big five". The initial focus of the subject was almost exclusively on a kind of zoological classifica-