Advances in Mathematics 317 (2017) 526–552

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Copies of the random graph

Miloš S. Kurilić [a,*], Stevo Todorčević [b,c]

[a] Department of Mathematics and Informatics, Faculty of Science, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia
[b] Institut de Mathématique de Jussieu (UMR 7586), Case 247, 4 Place Jussieu, 75252 Paris Cedex, France
[c] Department of Mathematics, University of Toronto, Toronto, M5S 2E4, Canada

ARTICLE INFO

Article history:
Received 10 October 2014
Received in revised form 10 June 2017
Accepted 26 June 2017
Available online xxxx
Communicated by Slawomir J. Solecki

MSC:
05C80
03C15
03C50
03E40
06A06
20M20

Keywords:
Random graph
Isomorphic substructure
Self-embedding
Partial order
Right Green's preorder
Forcing

ABSTRACT

Let $\langle R, \sim\rangle$ be the Rado graph, $\mathrm{Emb}(R)$ the monoid of its self-embeddings, $\mathbb{P}(R) = \{f(R) : f \in \mathrm{Emb}(R)\}$ the set of copies of R contained in R, and $\mathcal{I}_R$ the ideal of subsets of R which do not contain a copy of R. We consider the poset $\langle \mathbb{P}(R), \subset\rangle$, the algebra $P(R)/\mathcal{I}_R$, and the inverse of the right Green's preorder on $\mathrm{Emb}(R)$, and show that these preorders are forcing equivalent to a two step iteration of the form $\mathbb{P} * \pi$, where the poset $\mathbb{P}$ is similar to the Sacks perfect set forcing: adds a generic real, has the $\aleph_0$-covering property and, hence, preserves ω_1, has the Sacks property and does not produce splitting reals, while π codes an ω-distributive forcing. Consequently, the Boolean completions of these four posets are isomorphic and the same holds for each countable graph containing a copy of the Rado graph.

* Corresponding author.
E-mail addresses: milos@dmi.uns.ac.rs (M.S. Kurilić), stevo@math.univ-paris-diderot.fr, stevo@math.toronto.edu (S. Todorčević).

http://dx.doi.org/10.1016/j.aim.2017.06.037
0001-8708/

1. Introduction

In this paper we continue the investigation of the partial orderings of the form $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$, where $\mathbb{X}$ is an ultrahomogeneous relational structure and $\mathbb{P}(\mathbb{X})$ the set of domains of substructures of $\mathbb{X}$ isomorphic to $\mathbb{X}$. In particular, if $\mathbb{X} = \langle X, \rho \rangle$ is a binary structure (that is $\rho \subset X \times X$), then $\mathbb{P}(\mathbb{X}) = \{A \subset X : \langle A, \rho_A \rangle \cong \langle X, \rho \rangle\}$, where $\rho_A = \rho \cap (A \times A)$. In the sequel, in order to simplify notation, instead of $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ we will write $\mathbb{P}(\mathbb{X})$ whenever the context admits.

This investigation is related to a coarse classification of relational structures. Namely, the conditions $\mathbb{P}(\mathbb{X}) = \mathbb{P}(\mathbb{Y})$, $\mathbb{P}(\mathbb{X}) \cong \mathbb{P}(\mathbb{Y})$, $\operatorname{sq} \mathbb{P}(\mathbb{X}) \cong \operatorname{sq} \mathbb{P}(\mathbb{Y})$ and $\operatorname{ro} \operatorname{sq} \mathbb{P}(\mathbb{X}) \cong \operatorname{ro} \operatorname{sq} \mathbb{P}(\mathbb{Y})$ (where $\operatorname{sq} \mathbb{P}$ denotes the separative quotient of a partial order $\mathbb{P}$ and $\operatorname{ro} \operatorname{sq} \mathbb{P}$ its Boolean completion) define different equivalence relations ("similarities") on the class of relational structures and their interplay with the similarities defined by the conditions $\mathbb{X} = \mathbb{Y}$, $\mathbb{X} \cong \mathbb{Y}$ and $\mathbb{X} \rightleftarrows \mathbb{Y}$ (equimorphism) was considered in [11]. It turns out that the similarity defined by the condition $\operatorname{ro} \operatorname{sq} \mathbb{P}(\mathbb{X}) \cong \operatorname{ro} \operatorname{sq} \mathbb{P}(\mathbb{Y})$ is implied by all the similarities listed above and, thus, provides the coarsest among the mentioned classifications of relational structures. Since the posets of copies are always homogeneous, the condition $\operatorname{ro} \operatorname{sq} \mathbb{P}(\mathbb{X}) \cong \operatorname{ro} \operatorname{sq} \mathbb{P}(\mathbb{Y})$ is equivalent to the forcing equivalence of the posets $\mathbb{P}(\mathbb{X})$ and $\mathbb{P}(\mathbb{Y})$ (we will write $\mathbb{P}(\mathbb{X}) \equiv \mathbb{P}(\mathbb{Y})$) and, for convenience, we will exploit this fact using the tools of set-theoretic forcing in our proofs.

This paper can also be regarded as a part of the investigation of the quotient algebras of the form $P(\omega)/\mathcal{I}$, where $\mathcal{I}$ is an ideal on ω. Namely, by [8], if $\mathbb{X}$ is a countable indivisible[1] structure with domain ω, then the collection $\mathcal{I}_{\mathbb{X}}$ of subsets of ω which do not contain a copy of $\mathbb{X}$ is either the ideal of finite sets or a co-analytic tall ideal and the poset $\operatorname{sq} \mathbb{P}(\mathbb{X})$ is isomorphic to a dense subset of $(P(\omega)/\mathcal{I}_{\mathbb{X}})^+$, which implies $\operatorname{ro} \operatorname{sq} \mathbb{P}(\mathbb{X}) \cong \operatorname{ro}(P(\omega)/\mathcal{I}_{\mathbb{X}})^+$. So, since the structure considered in this paper, the Rado graph, $\langle R, \sim \rangle$, is indivisible, our results can be regarded as statements concerning the forcing related properties of the corresponding quotient algebra. Namely, if we call a graph *scattered* if it does not contain a copy of the Rado graph, and if $\mathcal{I}_R$ denotes the ideal of scattered subgraphs of R, then

$$\operatorname{ro} \operatorname{sq} \mathbb{P}(R) = \operatorname{ro}((P(R)/\mathcal{I}_R)^+).$$

As a consequence of the main result of [14] we have the following statement describing the forcing related properties of the poset of copies of the rational line, $\mathbb{Q}$, and the corresponding quotient $P(\mathbb{Q})/\operatorname{Scatt}$, where Scatt denotes the ideal of scattered suborders of $\mathbb{Q}$. Namely, if $\mathbb{S}$ denotes the Sacks perfect set forcing and $\operatorname{sh}(\mathbb{S})$ the size of the continuum in the Sacks extension, then we have

[1] A relational structure $\mathbb{X} = \langle X, \rho \rangle$ is *indivisible* (resp. *strongly indivisible*) iff for each partition of its domain X into two pieces one of them contains a copy of $\mathbb{X}$ (resp. one of them is a copy of $\mathbb{X}$). It is easy to see that the rational line, $\mathbb{Q}$, is an indivisible structure which is not strongly indivisible.