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Riemann–Roch theory for graph orientations

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ABSTRACT

We develop a new framework for investigating linear equivalence of divisors on graphs using a generalization of Gioan's cycle–cocycle reversal system for partial orientations. An oriented version of Dhar's burning algorithm is introduced and employed in the study of acyclicity for partial orientations. We then show that the Baker–Norine rank of a partially orientable divisor is one less than the minimum number of directed paths which need to be reversed in the generalized cycle–cocycle reversal system to produce an acyclic partial orientation. These results are applied in providing new proofs of the Riemann–Roch theorem for graphs as well as Luo's topological characterization of rank-determining sets. We prove that the max-flow min-cut theorem is equivalent to the Euler characteristic description of orientable divisors and extend this characterization to the setting of partial orientations. Furthermore, we demonstrate that $Pic^{g-1}(G)$ is canonically isomorphic as a $Pic^0(G)$-torsor to the equivalence classes of full orientations in the cycle–cocycle reversal system acted on by directed path reversals. Efficient algorithms for computing break divisors and constructing partial orientations are presented.

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Contents

1. Introduction

Baker and Norine [5] introduced a combinatorial Riemann–Roch theorem for graphs analogous to the classical statement for Riemann surfaces. For proving the theorem, they employed chip-firing, a deceptively simple game on graphs with connections to various areas of mathematics. Given a graph G, we define a *configuration of chips* D on G as a function from the vertices to the integers. A vertex v *fires* by sending a chip to each of its neighbors, losing its degree number of chips in the process. If we take D to be a vector, firing the vertex v_i precisely corresponds to subtracting the ith column of the Laplacian matrix from D. In this way we may view chip-firing as a combinatorial language for describing the integer translates of the lattice generated by the columns of the Laplacian matrix, e.g. [2,4].

Reinterpreting chip configurations as *divisors*, we say that two divisors are *linearly equivalent* if one can be obtained from the other by a sequence of chip-firing moves, and a divisor is *effective* if each vertex has a nonnegative number of chips. Baker and Norine define the *rank* of a divisor, denoted $r(D)$, to be one less than the minimum number of chips which need to be removed so that D is no longer equivalent to an effective divisor. Defining the *canonical divisor* K to have values $K(v) = \deg(v) - 2$, the *genus* of G to be $g = |E(G)| - |V(G)| + 1$, and the *degree* $\deg(D)$ of a divisor D to be the total number of chips in D, they prove the Riemann–Roch formula for graphs:

Theorem 1.1 *(Baker–Norine [5]).*

$$r(D) - r(K - D) = \deg(D) - g + 1.$$

Baker and Norine's proof depends in a crucial way on the theory of *q-reduced divisors*, known elsewhere as *G-parking functions* or *superstable configurations* [13,35]. A divisor D is said to be *q-reduced* if (i) $D(v) \geq 0$ for all $v \neq q$, and (ii) for any non-empty subset $A \subset V(G) \setminus \{q\}$, firing the set A causes some vertex in A to go into debt, i.e., to have a negative number of chips. They show that every divisor D is linearly equivalent to a unique q-reduced divisor D', and $r(D) \geq 0$ if and only if D' is effective. We note