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Explicit formulas and recurrence relations for higher order Eulerian polynomials

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Abstract

In the paper, the authors present explicit formulas, nonlinear ordinary differential equations, and recurrence relations for Eulerian polynomials, higher order Eulerian polynomials, and their generating functions in terms of the Stirling numbers of the second kind.

Keywords: Explicit formula; Nonlinear ordinary differential equation; Recurrence relation; Eulerian polynomial; Higher order Eulerian polynomial; Generating function; Stirling number of the second kind

1. Introduction

In [6], Kims stated that Eulerian polynomials $A_n(t)$ for $n \ge 0$ can be generated by

$$\frac{1-t}{e^{x(t-1)}-t} = \sum_{n=0}^{\infty} A_n(t)\frac{x^n}{n!}, \quad t \ne 1$$

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and that higher order Eulerian polynomials $A_n^{(\alpha)}(t)$ for integers $n \ge 0$ and real numbers $\alpha > 0$ can be generated by

$$\left[\frac{1-t}{e^{x(t-1)}-t}\right]^\alpha = \sum_{n=0}^\infty A_n^{(\alpha)}(t)\frac{x^n}{n!}, \quad t \ne 1.$$

This generation of $A_n(t)$ is same as the one in [1, p. 2], but different from the one

$$\frac{1-u}{e^{t(u-1)}-u} = 1 + \sum_{n=1}^\infty \frac{A_n(u)}{u}\frac{t^n}{n!}$$

in [3, p. 244, Eq. [5j]].

In [6, Theorem 1], Kims established inductively and recurrently that the generating function

$$F(t,x) = \frac{1}{e^{x(t-1)}-t}, \quad t \ne 1$$

satisfies the nonlinear ordinary differential equation

$$\frac{\partial^N F(t,x)}{\partial x^N} = (1-t)^N \sum_{i=1}^{N+1} a_{i-1}(N,t)F^i(t,x), \quad N \in \{0\}\cup\mathbb{N}, \tag{1.1}$$

where

$$a_0(N,t) = a_0(N-1,t) = \cdots = a_0(1,t) = a_0(0,t) = 1 \tag{1.2}$$

and

$$a_i(N,t) = it\sum_{j=0}^{N-i}(i+1)^j a_{i-1}(N-j-1,t), \quad 1 \le j \le N. \tag{1.3}$$

In [6, Theorems 2 and 3], Kims presented that

$$A_{n+N}(t) = (1-t)^{N+1}\sum_{i=1}^{N+1} a_{i-1}(N,t)\frac{A_n^{(i)}(t)}{(1-t)^i}$$

and

$$\sum_{j=0}^\infty t^j(j+1)^{n+N} = \frac{1}{(1-t)^n}\sum_{i=1}^{N+1} a_{i-1}(N,t)\frac{A_n^{(i)}(t)}{(1-t)^i}$$

for $n, N \in \{0\}\cup\mathbb{N}$. From (1.2) and (1.3), Kims derived inductively that

$$\begin{aligned} a_i(N,t) &= i!t^i \sum_{j_{i-1}=0}^{N-i} \sum_{j_{i-2}=0}^{N-j_{i-1}-i} \cdots \sum_{j_1=0}^{N-j_{i-1}-\cdots-j_2-i} (i+1)^{j_{i-1}} \\ &\quad \times i^{j_{i-2}} \cdots 3^{j_1}\left(2^{N-j_{i-1}-j_{i-2}-\cdots-j_1-i+1}-1\right) \end{aligned} \tag{1.4}$$

for $1 \le i \le N$.

It is clear that the above formulas (1.3) and (1.4) for $a_i(N,t)$ cannot be computed easily either by hand or by computer software. Can one find a simple expression for the quantities $a_i(N,t)$?

It is common knowledge that the Stirling numbers of the second kind $S(n,k)$ for $n \ge k \ge 0$ are important in combinatorial analysis, theory of special functions, number theory, and the like.

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