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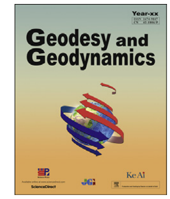

The ellipsoidal corrections for boundary value problems of deflection of the vertical with ellipsoid boundary

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The boundary value problem of deflections of vertical with ellipsoid boundary is studied in the paper. Based on spherical harmonic series, the ellipsoidal corrections for the boundary value problem are derived so that it can be well solved. In addition, an imitation arithmetic is given for examining the accuracies of solutions for the boundary value problem as well as its spherical approximation problem, and the computational results illustrate that the boundary value problem has higher accuracy than its spherical approximation problem if deflection of the vertical are measured on geoid.

1. Introduction

Although deflections of the vertical are important magnitudes in geodesy, their application values were not exploited enough in the past because of cost and time consuming in astro-geodetic techniques. Since deflections of the vertical are mainly computed from the gravity anomaly by Vening-Meinesz formula [1], they are usually considered as the derived magnitudes in geodesy. However, with the development of transportable photographic zenith cameras, astro-geodetic observation techniques are improved largely [2–6] so that deflections of the vertical can be rapidly measured, and the measurement accuracy can reach to 0.07″. Accompanied with applications of astro-geodetic observation techniques, some researches about applications of deflections of the verticals are proposed, e.g., Hirt et al. [7].

Since the disturbing potential can be expressed from deflections of the vertical by inverse Vening-Meinesz formula, rapid astro-geodetic measurement techniques will have potential applicable values in determining the gravity field. Interestingly, although it is cost and time consuming to establish a dense measurement net of deflections of the vertical, deflections of the vertical can be computed from satellite altimetry data at the cross-over points of the orbits. In fact, data of deflections of the vertical from altimetry have been used to solve the parameters of the gravity field of the earth [8,9].

On the other hand, for geodetic boundary value problems (BVPs) such as Dirichlet, Neumann and Stokes' ones, there have been a lot of studies and discussions about solving methods in the case of ellipsoid boundary [1,10–18]. Hence, it is helpful for consummating geodetic BVP theories to introduce and derive the ellipsoidal corrections of BVP of deflection of the vertical with ellipsoid boundary.

Introducing the disturbing potential T, we have $\xi = \frac{1}{R\gamma}\frac{\partial T}{\partial \theta}$ and $\eta = -\frac{1}{R\gamma \sin\theta}\frac{\partial T}{\partial \lambda}$, where R is the average radius of the earth, γ the normal gravity, (r, θ, λ) the spherical coordinates, θ co-latitude, λ longitude, and ξ , η deflections of the vertical. If ξ, η can be measured on ocean surface or geoid, i.e., $\frac{\partial T}{\partial \theta}$ and $-\frac{1}{\sin\theta}\frac{\partial T}{\partial \lambda}$ can be

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given in advance, the following boundary value problem can be constituted

$$\begin{cases} \Delta T = 0, \quad \text{outside } \Sigma \\ \left.\dfrac{\partial T}{\partial \theta}\right|_{\Sigma} = \alpha \\ \left.\dfrac{1}{\sin\theta}\dfrac{\partial T}{\partial \lambda}\right|_{\Sigma} = -\beta \\ T = O(r^{-1}), \quad \text{at infinity} \end{cases} \tag{1}$$

where $\alpha = R\gamma\xi$ and $\beta = R\gamma\eta$ are all known on Σ, and Σ is geoid. Since geoid is very close to the reference ellipsoid, Σ can be chosen as the reference ellipsoid under the accuracy of $O(T^2)$. Later on, Σ is always chosen as the reference ellipsoid, i.e., $\Sigma : \frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1$, where a and b are semi-axes of the reference ellipsoid, and Eq. (1) is called DV-BVP (DV: deflection of the vertical) with the ellipsoid boundary.

In order to make Eq. (1) solvable, the corresponding spherical approximation is introduced, i.e., after Σ in Eq. (1) is approximately chosen as the average sphere of the earth, we have

$$\begin{cases} \Delta T = 0, \quad \text{outside } \{r = R\} \\ \left.\dfrac{\partial T}{\partial \theta}\right|_{r=R} = \alpha \\ \left.\dfrac{1}{\sin\theta}\dfrac{\partial T}{\partial \lambda}\right|_{r=R} = -\beta \\ T = O(r^{-1}), \quad \text{at infinity} \end{cases} \tag{2}$$

The integral solution of Eq. (2) can be given by inverse Vening-Meinesz formula [8,9]. Furthermore, the spherical harmonic coefficients of the disturbing potential T can be also obtained from Eq. (2). Hence, Eq. (2) is a complete solvable from mathematical point of view. Later on, Eq. (2) is called DV-BVP under the spherical approximation.

Although Eq. (2) is solvable, it still is a simplified form of Eq. (1). In order to raise the accuracy of solving the disturbing potential, it is necessary to solve Eq. (1) directly. In fact, our aim is to solve DV-BVP with the ellipsoid boundary, i.e., to find an arithmetic method for the spherical harmonic coefficients of T from Eq. (1). Since the boundary Σ in Eq. (1) is the reference ellipsoid, it is more difficult to solve Eq. (1) compared with Eq. (2).

For BVP with the ellipsoid boundary, the general solving approach is to find the solution with accuracy of $O(\varepsilon^4 \cdot T)$ after introducing the ellipsoidal corrections, where $\varepsilon^2 = \frac{a^2-b^2}{b^2}$. As for derivation of the ellipsoidal corrections, there are two main approaches. The first is: to introduce the ellipsoidal harmonic series of the disturbing potential T to solve BVP with the ellipsoid boundary and then to compute the spherical harmonic coefficients of T by transformation formula between the spherical harmonic series and ellipsoidal ones [11,12,19,20]; and the second is: to reduce the ellipsoid Σ into the average sphere $\{r = R\}$ of the earth by making use of Taylor's expansion for T. Since a lot of computations about the ellipsoidal harmonic series are included in the first approach, which can make the derivation complicated, the second approach for deriving the ellipsoidal corrections is adopted in the paper, i.e., our aim is to obtain the solution of Eq. (2) under the condition that Eq. (1) has been solved.

2. Preliminaries

At first, the spherical harmonic expansion of T can be written as

$$T = \sum_{n=1}^{\infty}\sum_{m=0}^{n} \frac{R^n}{r^{n+1}} \overline{P}_{nm}(\cos\theta)(a_{nm}\cos m\lambda + b_{nm}\sin m\lambda) \tag{3}$$

where $R = \sqrt[3]{a^2 b}$ is the average radius of the earth, and

$$\overline{P}_{n,m}(x) = \sqrt{\frac{2n+1}{2(1+\delta_{m,0})}}\sqrt{\frac{(n-m)!}{(n+m)!}} P_{n,m}(x) \tag{4}$$

where $P_{n,m}(x) = (1-x^2)^{\frac{m}{2}} P_n^{(m)}(x)$, $P_n(x) = \frac{(-1)^n}{2^n n!}\frac{\mathrm{d}^n[(1-x^2)^n]}{\mathrm{d}x^n}$ and $\delta_{mk} = \begin{cases} 0, & m \neq k \\ 1, & m = k \end{cases}$. In order to make the following derivation simple, the notation that $\overline{P}_{nm}(x) = 0$ is always used when $n < m$.

Introducing $c_{nm} = \begin{cases} a_{nm}, & m \geq 0 \\ b_{nm}, & m < 0 \end{cases}$ and

$$W_{nm}(\theta, \lambda) = \begin{cases} \overline{P}_{nm}(\cos\theta)\cos m\lambda, & m \geq 0 \\ \overline{P}_{nm}(\cos\theta)\sin m\lambda, & m < 0 \end{cases} \tag{5}$$

the spherical harmonic expansion of T can be rewritten as

$$T = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \frac{R^n}{r^{n+1}} c_{nm} W_{nm}(\theta, \lambda) \tag{6}$$

Again introducing the vector-value functions

$$\mathbf{v}_{nm}(\theta, \lambda) = \left(-\frac{\partial W_{nm}(\theta,\lambda)}{\partial\theta}, \frac{1}{\sin\theta}\frac{\partial W_{nm}(\theta,\lambda)}{\partial\lambda}\right) \tag{7}$$

it can be easily proved that

$$\iint_{\sigma} \mathbf{v}_{nm}(\theta,\lambda)\cdot\mathbf{v}^{\mathrm{T}}_{n'm'}(\theta,\lambda)\mathrm{d}\sigma = n(n+1)\pi\delta_{nn'}\delta_{mm'} \tag{8}$$

where σ is the unit sphere.

After introducing the above notations, it can be concluded from Eq. (2)

$$(\alpha, \beta) = -\sum_{n=1}^{\infty}\sum_{m=-n}^{n} \frac{c_{nm}}{R}\mathbf{v}_{nm}(\theta, \lambda) \tag{9}$$

Letting

$$C_{nm} = -\frac{R}{\pi n(n+1)}\iint_{\sigma}(\alpha, \beta)\cdot\mathbf{v}^{\mathrm{T}}_{nm}(\theta,\lambda)\mathrm{d}\sigma \tag{10}$$

it can be seen that C_{nm} are the spherical harmonic coefficients of T in Eq. (2). This means that if α and β are known on the sphere $\{r = R\}$ in Eq. (2), the spherical harmonic coefficients of T can be easily computed from Eq. (10).

3. The solution of DV-BVP with the ellipsoid boundary

In the following derivations, only the terms equivalent to $O(\varepsilon^2)$ are retained since the magnitudes above $O(T\cdot\varepsilon^4)$ are always neglected in solving Eq. (1). Letting Q be a point on Σ and using Q_1 to denote the corresponding point of Q on $\{r = R\}$ along the radial direction, we have

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