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On approximations of the basic equations of terrestrial mantle convection used in published literature

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ABSTRACT

Previous studies have suggested some approximations for the basic equations of terrestrial mantle convection. The approximations are based on five dimensionless parameters—*M* (Much number), *Pr* (Prandtl number), *Di* (dissipation number), *Co* (Compressibility number; the ratio of dissipation number to the Grüneisen number), and υ (fraction of density change due to thermal expansion). These approximations are given by: (i) $M^2Pr \to 0$ for the anelastic liquid approximation (ALA), (ii) $M^2Pr \to 0$ and $\upsilon \to 0$ for the truncated anelastic liquid approximation (TALA), (iii) $M^2Pr \to 0$, $\upsilon \to 0$, and $Co \to 0$ for the extended Boussinesq approximation (EBA), (iv) $M^2Pr \to 0$, $\upsilon \to 0$, and $Di/Ra \to 0$ for the superadiabatic Boussinesq approximation (SBA), and (v) $M^2Pr \to 0$, $\upsilon \to 0$, $Co \to 0$, and $Di \to 0$ for the Boussinesq approximation (BA). This study suggests the use of five dimensionless parameters, namely, *M*, *Pr*, *Di*, *Co*, and *Ra* (Rayleigh number), to reduce the number of approximations to four: (I) $M^2Pr \to 0$ for the ALA, (II) $M^2Pr \to 0$ and $Co \to 0$ for the EBA, (III) $M^2Pr \to 0$ and $Di/Ra \to 0$ for the SBA, and (IV) $M^2Pr \to 0$, $Co \to 0$, and $Di \to 0$ for the BA. This is because υ is simply defined by $\upsilon = M^2PrRa/Co$ and is automatically approximated to 0 when $M^2Pr \to 0$. In other words, approximations of ALA and TALA can be unified because they represent the same sense physically. This conclusion is valid for mantle convection in the present Earth whose $Ra \sim O(10^7)$ is approximately one order smaller than the threshold Rayleigh number, $Ra_{thr} = Co/(M^2Pr) \sim O(10^8)$.

1. Introduction

Theoretical and numerical (computational) studies on the mantle dynamics use appropriate approximations and simplifications of the basic equations that govern mantle convection depending on the research problem under consideration. Schubert et al. (2001) provided a detailed derivation of the basic equations governing mantle convection, i.e., the conservation equations of mass, momentum, and energy based on several previous studies (Ita and King, 1994; Jarvis and McKenzie, 1980; McKenzie et al., 1974; Schmeling, 1989; Schmeling and Jacoby, 1981; Tackley, 1996; Zhang and Yuen, 1996). Five approximation methods have been suggested to simplify these basic equations (Table 1): anelastic liquid approximation (ALA) (McKenzie et al., 1974; Zhang and Yuen, 1996), truncated anelastic liquid approximation (TALA) (Ita and King, 1994; Jarvis and McKenzie, 1980; Steinbach et al., 1989), extended Boussinesq approximation (EBA) (Christensen and Yuen, 1985), and Oberbeck–Boussinesq approximation (hereafter, simply called the "Boussinesq approximation"; BA) (Boussinesq, 1903; Oberbeck, 1879). As an intermediate approximation between the EBA and BA, Trubitsyn and Trubitsyn (2015) introduced the superadiabatic Boussinesq approximation (SBA) in which the adiabatic heating term in the conservation equation of energy is retained, whereas the viscous dissipation term is ignored under the EBA.

The purpose of this brief paper is to show that the number of approximation methods for mantle convection can be reduced to four by considering the rearrangement of the dimensionless parameters. The forms of equations and symbols for physical quantities and parameters shown in this paper are based on published work by Schubert et al. (2001) and others (Christensen and Yuen, 1985; Ita and King, 1994; Jarvis and McKenzie, 1980; McKenzie et al., 1974; Schmeling, 1989; Schmeling and Jacoby, 1981; Zhang and Yuen, 1996).

2. Dimensionless parameters and the reference states

2.1. Non-dimensionalization

In the derivation of the basic equations governing terrestrial mantle convection, the length *L*, the differential operator ∇, thermal diffusion time *t*, velocity $\boldsymbol{v}$, pressure *p* (or *P*), stress σ, temper-

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Table 1
Approximations for the basic equations of mantle convection in previous literature and this paper.

Step#	Name of approximations	Approximations
Previous literature (5 steps)		
i	Anelastic liquid approximation (ALA)	$M^2Pr \to 0$
ii	Truncated anelastic liquid approximation (TALA)	$M^2Pr \to 0$, $\upsilon \to 0$
iii	Extended Boussinesq approximation (EBA)	$M^2Pr \to 0$, $\upsilon \to 0$, $Co \to 0$
iv	Superadiabatic Boussinesq approximation (SBA)	$M^2Pr \to 0$, $\upsilon \to 0$,$Di/Ra \to 0$
v	Boussinesq approximation (BA)	$M^2Pr \to 0$, $\upsilon \to 0$, $Co \to 0$, $Di \to 0$
This paper (4 steps)		
I	Anelastic liquid approximation (ALA)	$M^2Pr \sim \upsilon \to 0$
II	Extended Boussinesq approximation (EBA)	$M^2Pr \sim \upsilon \to 0$, $Co \to 0$
III	Superadiabatic Boussinesq approximation (SBA)	$M^2Pr \sim \upsilon \to 0$, $Di/Ra \to 0$
IV	Boussinesq approximation (BA)	$M^2Pr \sim \upsilon \to 0$, $Co \to 0$, $Di \to 0$

ature T, and gravitational potential ψ were non-dimensionalized as follows (e.g., Schubert et al., 2001):

$$L' = \frac{1}{r_{top}}L,\quad \nabla' = r_{top}\nabla,\quad t' = \frac{\kappa_0}{r_{top}^2}t,\quad \boldsymbol{v}' = \frac{r_{top}}{\kappa_0}\boldsymbol{v},$$
$$p' = \frac{r_{top}^2}{\eta_0\kappa_0}p,\quad \boldsymbol{\sigma}' = \frac{r_{top}^2}{\eta_0\kappa_0}\boldsymbol{\sigma},\quad T' = \frac{1}{\Delta T}(T + T_{top}), \tag{1}$$
$$\Psi' = \frac{r_{top}^3\rho_0 g_0}{\eta_0\kappa_0}\Psi,$$

where r_{top} denotes the radius of the planet, T_{top} is the absolute temperature at the planet's surface, and ΔT is the temperature difference between the top and bottom surfaces of the mantle. Other quantities such as the density ρ, coefficient of thermal expansion α, gravitational acceleration g (positive for upward direction), thermal diffusivity κ, viscosity η, specific heat at a constant pressure or volume (c_p and c_v), and isothermal or adiabatic bulk modulus (K_T and K_a) were dimensionalized by their reference values, e.g., $\rho' = \rho/\rho_0$. The subscript 0 indicates reference values (Table 2), and the variables with primes represent the dimensionless quantities.

In this study, the physical quantities were normalized by the radius of the planet, r_{top}, along the spherical polar coordinates (r, θ, ϕ). The basic equations were non-dimensionalized by r_{top} following the dimensionless factors of Eq. (1), while the dimensionless parameters were scaled by the thickness of the terrestrial mantle, b, to ensure consistency between the models in Cartesian geometry and those in spherical-shell geometry. The ratio of the thickness of the terrestrial mantle to the radius of the planet ξ is defined as

$$\xi \equiv \frac{b}{r_{top}}, \tag{2}$$

while the relationship between the dimensionless parameter β scaled by b and that scaled by r_{top} is given by

$$\beta(r_{top}^n) = \beta(b^n)\cdot\xi^{-n}, \tag{3}$$

where n is the power-law integer. If the dimensionless radii of the top surface boundary of the planet and the center of planet are fixed at 1 and 0 respectively, then $0 < \xi \le 1$

2.2. Reference states and the deviations

Using the dimensionless factors of Eq. (1), the dimensionless forms of the thermodynamically stabilized reference states of pressure (i.e., hydrostatic/lithostatic pressure) $\bar{P}'$, temperature (i.e., adiabatic temperature) $\bar{T}'_a$, and density (i.e., stratified adiabatic density in gravity) $\bar{\rho}'_a$ are expressed by

$$\frac{d\bar{P}'}{dr'} = -\frac{Co}{M^2Pr}\xi^{-3}\bar{\rho}'_a\bar{g}',$$
$$\frac{d\bar{T}'_a}{dr'} = -Di\xi^{-1}\frac{\bar{\alpha}'\bar{g}'}{\bar{c}'_p}\bar{T}'_a, \tag{4}$$
$$\frac{d\bar{\rho}'_a}{dr'} = -Co\xi^{-1}\bar{\rho}'_a,$$

and by integrating, the followings equation are obtained,

$$\bar{P}' = \bar{P}'_s + \frac{Co}{M^2Pr}\xi^{-3}\bar{\rho}'_a\bar{g}'(r'_{top} - r'),$$
$$\bar{T}'_a = \bar{T}'_{as}\exp\left[Di\xi^{-1}\frac{\bar{\alpha}\bar{g}}{\bar{c}_p}(r'_{top} - r')\right], \tag{5}$$
$$\bar{\rho}'_a = \bar{\rho}'_{as}\exp\left[Co\xi^{-1}(r'_{top} - r')\right],$$

where $\bar{\alpha}$ is the reference coefficient of thermal expansion, $\bar{c}_p$ is the reference specific heat at a constant pressure, and the subscript s indicates the reference value at the planet's surface. The dimensionless parameters M, Pr, Gr, Di, and Co are the Mach number (the ratio of thermal diffusive velocity to the sound velocity), the Prandtl number (the ratio of momentum diffusion to the thermal diffusion), the Grüneisen number, the dissipation number, and the compressibility number, respectively. They are given by

$$M \equiv \frac{\kappa_0/b}{(K_{T0}/\rho_0)^{1/2}} = \left(\frac{\rho_0\kappa_0^2}{b^2K_{T0}}\right)^{\frac{1}{2}},\quad Pr \equiv \frac{\eta_0}{\rho_0\kappa_0},\quad Gr \equiv \frac{\alpha_0K_{T0}}{\rho_0c_{v0}} = \frac{\alpha_0K_{a0}}{\rho_0c_{p0}},\quad Di \equiv \frac{\alpha_0g_0b}{c_{p0}},$$
$$Co \equiv \frac{Di}{Gr} = \frac{g_0b\rho_0c_{v0}}{K_{T0}c_{p0}} = \frac{g_0b\rho_0c_{p0}}{K_{a0}c_{p0}}. \tag{6}$$

Here, Di and Gr are related to the compressibility of the terrestrial mantle (Birch, 1952). Note that Eq. (5) is valid if Gr and Co are assumed to be constant and do not depend on the density. The depth-dependence of Gr can be referred in Steinbach et al. (1989) and Balachandar et al. (1992). Moreover, $\bar{g}'$ is assumed constant and does not depend on the radius. To determine the radial change of $\bar{g}'$ in a real spherical-shell geometry, refer Schubert et al. (2001).

The dynamic pressure (or nonhydrostatic pressure) p and the potential temperature T deviate from their reference state owing to convective motion. They are expressed as

$$p' = P' - \bar{P}',$$
$$T' = T'_a - \bar{T}'_a. \tag{7}$$

Using the dimensionless factors in Eq. (1), the deviation of density from the reference state owing to convective motion is expressed as follows (e.g., Schubert et al., 2001)

$$\rho' = \bar{\rho}\left(1 + \frac{1}{\bar{K}'_T}p'M^2Pr\xi^2 - \bar{\alpha}'T'\upsilon\right), \tag{8}$$

where υ is the fraction of density change due to thermal expansion

$$\upsilon \equiv \alpha_0\Delta T, \tag{9}$$

and from Eq. (6),

$$M^2Pr = \frac{\eta_0\kappa_0}{K_{T0}b^2}. \tag{10}$$

In the following section, the basic equations of terrestrial mantle convection have been reconsidered under the approximation methods suggested in the published literature. For simplicity, the primes representing dimensionless quantities have been omitted hereafter.

3. Approximations for the basic equations of mantle convection

3.1. Anelastic liquid approximation (ALA)

In the convection of a compressible fluid with a low Prandtl number, the Mach number is approximated to zero ($M \to 0$) under the assumption that the short-wavelength phenomena by