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How long does it take for aquifer recharge or aquifer discharge processes to reach steady state?

Matthew J. Simpson [a,b,*], Farhad Jazaei [c], T. Prabhakar Clement [c]

[a] Mathematical Sciences, Queensland University of Technology, GPO Box 2434, Brisbane, Queensland 4001, Australia
[b] Tissue Repair and Regeneration Program, Institute of Health and Biomedical Innovation, Queensland University of Technology, Brisbane, Australia
[c] Department of Civil Engineering, Auburn University, Auburn, AL 36849, USA

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SUMMARY

Groundwater flow models are usually characterized as being either transient flow models or steady state flow models. Given that steady state groundwater flow conditions arise as a long time asymptotic limit of a particular transient response, it is natural for us to seek a finite estimate of the amount of time required for a particular transient flow problem to effectively reach steady state. Here, we introduce the concept of *mean action time* (MAT) to address a fundamental question: how long does it take for a groundwater recharge process or discharge processes to effectively reach steady state? This concept relies on identifying a cumulative distribution function, $F(t;x)$, which varies from $F(0;x) = 0$ to $F(t;x) \to 1^-$ as $t \to \infty$, thereby providing us with a measurement of the progress of the system towards steady state. The MAT corresponds to the mean of the associated probability density function $f(t;x) = \mathrm{d}F/\mathrm{d}t$, and we demonstrate that this framework provides useful analytical insight by explicitly showing how the MAT depends on the parameters in the model and the geometry of the problem. Additional theoretical results relating to the variance of $f(t;x)$, known as the *variance of action time* (VAT), are also presented. To test our theoretical predictions we include measurements from a laboratory-scale experiment describing flow through a homogeneous porous medium. The laboratory data confirms that the theoretical MAT predictions are in good agreement with measurements from the physical model.

1. Introduction

Groundwater flow systems, and the corresponding models used to study these systems, are typically characterized as being either transient or steady state (Remson et al., 1971; Bear, 1972; Clement et al., 1994; Haitjema, 1995; Strack, 1989; Wang and Anderson, 1982; Zheng and Bennett, 2002). This characterization is useful since the mathematical and computational techniques required to solve steady state groundwater flow models are generally much simpler than those required to solve transient groundwater flow models. Given that steady flow conditions correspond to the long time asymptotic limit of a transient response (Wang and Anderson, 1982, pp. 76–77; Haitjema, 1995, pp. 158–159) it is relevant to develop tools that can be used to estimate the amount of time required for a particular transient flow problem to effectively reach steady state. In the heat and mass transfer literature such a time is called a *critical time* (Hickson et al., 2009a,b, 2011).

A schematic diagram of a groundwater recharge problem is outlined in Fig. 1(a) for an aquifer of length L. The aquifer is bounded by two rivers. River one, at $x = 0$, at river stage h_1, and river two, at $x = L$, at river stage h_2. The hypothetical phreatic surface without recharge is indicated by the curve marked $t = 0$. We consider initiating a transient response in the groundwater flow system by applying spatially uniform recharge at rate R. The result of applying this recharge is that the amount of water stored in the aquifer increases with time as the phreatic surface rises to reach the curve indicated by $t \to \infty$. This kind of scenario, where recharge is applied to an existing unconfined groundwater flow system, leads to an increase in the saturated depth corresponding to an increase in the amount of water stored in the aquifer. The details of how to design and operate such recharge systems have been described at length previously (Bouwer, 2002; Daher et al., 2011; Martín-Rosales et al., 2007; Pedretti et al., 2012; Vandenbohede and Van Houtte, 2012). The design of such recharge systems naturally leads to the following questions:

* Corresponding author at: Mathematical Sciences, Queensland University of Technology, GPO Box 2434, Brisbane, Queensland 4001, Australia. Tel.: +61 7 3138 5241; fax: +61 7 3138 2310.
E-mail address: matthew.simpson@qut.edu.au (M.J. Simpson).

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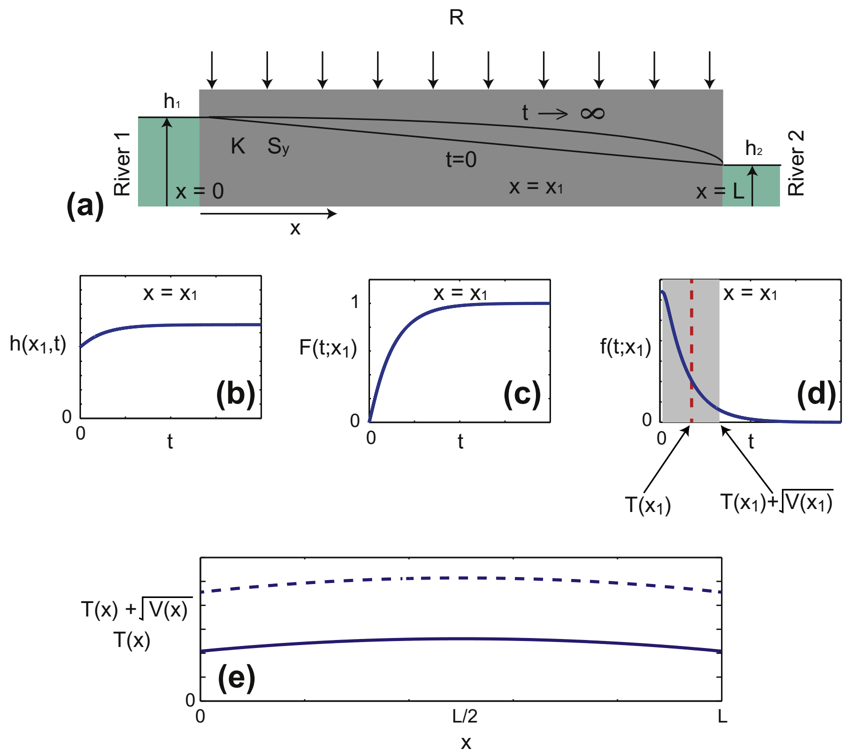

Fig. 1. (a) Schematic of an aquifer recharge process. The groundwater flow takes place on a one-dimensional domain, $0 \leqslant x \leqslant L$, and is assumed to correspond to a linearized, unconfined, Dupuit–Forchheimer description (Bear, 1972). The saturated depth at $x = 0$ (river 1) is $h(0,t) = h_1$. The saturated depth at $x = L$ (river 2) is $h(L,t) = h_2$. The schematic depicts a transition where the initial phreatic surface, indicated by $t = 0$, asymptotes to a new steady state, indicated by $t \to \infty$. This transition is associated with the application of uniform recharge, at rate R, for $t > 0$. (b) Schematic showing how the saturated thickness at a fixed location, $x = x_1$, in Fig. 1(a) varies with time, t. This schematic corresponds to a recharge transition since $h(x,t)$ increases with t. (c) For the schematic transition in (b) we show $F(t;x_1)$, which has the property that $F(0;x_1) = 0$ and $F(t;x_1) \to 1^-$ as $t \to \infty$. (d) For the schematic transition in (b) we plot $f(t;x_1)$, using Eq. (4). The mean of this probability density function is indicated in the red vertical (dotted) line, and corresponds to the MAT, $T(x_1)$. The variance of this probability density function is indicated with the gray shading, which corresponds to one standard deviation about the mean $T(x_1) \pm \sqrt{V(x_1)}$, as indicated. Profiles in (e) show $T(x)$ (solid) and $T(x) + \sqrt{V(x)}$ (dashed) at all locations $0 \leqslant x \leqslant L$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

(1) How long does it take for the volume of water stored in the aquifer to reach a maximum? (i.e. what is the critical time for this process?)
(2) How does this critical time depend on the parameters governing the flow processes and the geometry of the aquifer?

Strictly speaking, from a mathematical point of view, it takes an infinite amount of time for a transient response of a diffusive process to become steady (McNabb and Wake, 1991; McNabb, 1993). Clearly, this strict mathematical definition is impractical and it would be useful to have a quantitative framework to estimate a finite timescale that indicates when the time rate of change of water stored in the aquifer to effectively reach zero (Sophocleus, 2012; Walton, 2011). Developing a method of analysis that avoids the need for relying on numerical computation to answer these questions would be useful since it is not obvious how, for example, changing the properties of the porous medium or the geometry of the groundwater flow system would affect the time taken for the rate of change of water stored in the aquifer to effectively reach zero. Understanding this timescale may have several practical uses; for example, if we were to design an artificial recharge program it would be of interest to monitor the increase in storage in the aquifer with time and to have a criteria to indicate when the system would effectively reach steady state.

Previous attempts to characterize critical times for groundwater flow models have relied on using numerical experimentation (Buès and Oltean, 2000; Chang et al., 2011), laboratory-scale experimentation (Kim and Ann, 2001; Goswami and Clement, 2007; Chang and Clement, 2012; Simpson et al., 2003) or very simple mathematical definitions. One common mathematical approach is to define the critical time to be the amount of time taken for the transient solution to reach within ϵ% of the corresponding steady state value, where ϵ is some small user-defined tolerance (Hickson et al., 2011; Landman and McGuinness, 2000; Lu and Werner, 2013; Watson et al., 2010). Although insightful, there are certain difficulties associated with this definition, namely:

(1) this definition depends upon a subjective choice of ϵ,
(2) this definition requires the complete solution of the the transient groundwater flow problem, and
(3) this definition leads to a numerical framework that does not provide analytical insight into how the critical time varies with the parameters in the model.

In this work we introduce the concept of *mean action time* (MAT) which gives us a finite estimate of the amount of time required for a transient groundwater flow resposne to effectively reach steady state. The MAT was originally defined by McNabb and Wake as a tool to study linear heat transfer (McNabb and Wake, 1991; McNabb, 1993). Here we demonstrate how to extend this theory to analyze groundwater flow processes. We will show, in a general framework, that:

(1) the MAT gives us an objective finite estimate of the amount of time required for a transient response to effectively reach steady state,