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Periodicity of free subgroup numbers modulo prime powers

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ABSTRACT

We characterise when the sequence of free subgroup numbers of a finitely generated virtually free group is ultimately periodic modulo a given prime power.

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1. Introduction

For a finitely generated virtually free group Γ, denote by m_Γ the least common multiple of the orders of the finite subgroups in Γ and, for a positive integer λ, let $f_\lambda(\Gamma)$ be the number of free subgroups of index λm_Γ in Γ. In [4], the authors show, among other things, that the number $f_\lambda(\mathrm{PSL}_2(\mathbb{Z}))$ of free subgroups of index 6λ in the inhomogeneous modular group $\mathrm{PSL}_2(\mathbb{Z})$, considered as a sequence indexed by λ, is ultimately periodic modulo any fixed prime power p^α, if p is a prime number with $p \geq 5$. More precise results on the length of the period, and an explicit formula for the linear recurrence satisfied by these numbers modulo p^α are also provided in [4]. As is well known, ultimate periodicity of the sequence $\big(f_\lambda(\Gamma)\big)_{\lambda\geq 1}$ is equivalent to rationality of the corresponding generating function $F_\Gamma(z) = \sum_{\lambda\geq 0} f_{\lambda+1}(\Gamma) z^\lambda$.

The purpose of the present paper is to demonstrate that the periodicity phenomenon discovered in [4] holds in a much wider context, namely that of finitely generated virtually free groups. Indeed, our main result (Theorem 1) provides an explicit characterisation of all pairs (Γ, p^α), where Γ is a finitely generated virtually free group and p^α is a proper prime power, for which the sequence of free subgroup numbers of Γ is ultimately periodic modulo p^α. Roughly speaking, for "almost all" pairs (Γ, p) the sequence $\big(f_\lambda(\Gamma)\big)_{\lambda\geq 0}$ is ultimately periodic modulo p^α for all $\alpha \geq 1$, the only exception occurring when $p \mid m_\Gamma$ and $\mu_p(\Gamma) = 0$, where $\mu_p(\Gamma)$ is a certain invariant defined in (2.10) and discussed in the paragraph following that formula.

In order to further place our results into context, we point out that, for primes p dividing the constant m_Γ, an elaborate theory is presented in [9] for the behaviour of the arithmetic function $f_\lambda(\Gamma)$ modulo p. Recently, this theory has been supplemented by congruences modulo (essentially arbitrary) 2-powers and 3-powers for the number of free subgroups of finite index in lifts of the classical modular group; that is, amalgamated products of the form

$$\Gamma_\ell = C_{2\ell} \underset{C_\ell}{*} C_{3\ell}, \quad \ell \geq 1;$$

cf. Theorems 19 and 20 in [3, Sec. 8], and Section 16 in [5], in particular, [5, Thms. 49–52]. These results demonstrate a highly non-trivial behaviour of the sequences $\big(f_\lambda(\Gamma_\ell)\big)_{\lambda\geq 1}$ modulo powers of 2 if ℓ is odd (in which case $\mu_2(\Gamma_\ell) = 0$), and modulo powers of 3 for $3 \nmid \ell$ (in which case $\mu_3(\Gamma_\ell) = 0$). For instance, for the sequence $\big(f_\lambda = f_\lambda(\Gamma_1)\big)_{\lambda\geq 1}$ of free subgroup numbers of the group $\mathrm{PSL}_2(\mathbb{Z})$, one finds that:

(1) $f_\lambda \equiv -1 \pmod 3$ if, and only if, the 3-adic expansion of λ is an element of $\{0,2\}^*1$;
(2) $f_\lambda \equiv 1 \pmod 3$ if, and only if, the 3-adic expansion of λ is an element of

$$\{0,2\}^*100^* \cup \{0,2\}^*122^*;$$

(3) for all other λ, we have $f_\lambda \equiv 0 \pmod 3$;