Journal of Algebra 373 (2013) 426–438

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On supersimple groups ☆

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ARTICLE INFO

Article history:
Received 24 May 2012
Available online 31 October 2012
Communicated by E.I. Khukhro

MSC:
primary 03C45, 03C60
secondary 20F16, 20F24, 20F14

Keywords:
Model theory
Supersimple group
Just-infinite groups
Series with abelian or simple factors

ABSTRACT

We show that an infinite group having a supersimple theory has a finite series of definable subgroups whose factors are infinite and either virtually FC or virtually simple modulo a finite FC-centre. We deduce that a group which is type-definable in a supersimple theory has a finite series of relatively definable subgroups whose factors are either abelian or simple groups. In this decomposition, the non-abelian simple factors are unique up to isomorphism.

1. Introduction

Model theory is the study of *definable sets*, that is, the sets that are defined by a first order formula in a given language. It can be thought of as a generalisation of algebraic geometry where the objects under study are algebraic varieties, which are defined by systems of polynomial equations; in this case, the language considered is the language of fields which consists of the two function symbols $+$, $\times$ and the two constants 0 and 1. The existence of a notion of dimension on a class of definable sets strongly restricts the behavior of this class. In a linear algebraic group over an algebraically closed field, the Zariski dimension of an algebraic variety arises from the Zariski topology. A *supersimple group* is a group whose definable sets are equipped with a notion of dimension, the so-called *SU-rank*, arising from the logic topology and taking ordinal values. The SU-rank extends the Lascar U-rank of superstable groups and the Morley rank of groups of finite Morley rank. It is thus a far reaching generalisation of the Zariski dimension for linear algebraic groups. Examples of supersimple groups include:

☆ Thanks to Wagner for improving an earlier version of Theorem 5.3.

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0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2012.09.033

- finite groups;
- linear algebraic groups over algebraically closed fields;
- groups of finite Morley rank;
- abelian groups which are either divisible, or of bounded exponent;
- $\aleph_0$-stable groups;
- superstable groups;
- simple linear algebraic groups over pseudofinite fields;
- simple pseudofinite groups (*i.e.* abstractly simple groups having a pseudofinite theory).

On the other hand, the infinite cyclic group $\mathbf{Z}$, the non-abelian free groups on n generators and more generally the non-elementary hyperbolic groups are not supersimple (see Corollary 4.10).

In spite of the strong analogy between the U-rank and the SU-rank, the theory of supersimple groups is not nearly as developed as it is for its superstable analogues. For instance, Berline and Lascar [BL] have shown that every superstable group has a definable abelian subgroup of the same cardinality. Their original argument has been much simplified since and has shrunk to a few lines (see [Wag00, Remark 5.4.11]). Still, it is unknown whether a supersimple group G has an infinite abelian subgroup. If H is an abelian subgroup of G, then there exists a definable subgroup E of G which is finite-by-abelian and contains H (see [EJMR] or [Mil1]). Moreover, if one writes the SU-rank of G in the form $\omega^\alpha.n + \beta$ with $0 \leqslant \beta < \omega^\alpha$, then one can provide that the SU-rank of E is at least ω^α. So, asking whether G has an infinite abelian subgroup is equivalent with asking whether G has a definable finite-by-abelian subgroup of SU-rank at least ω^α. Finite-by-abelian groups are *FC groups*, *i.e.* groups whose every conjugacy class is finite. In [Bau], Baudisch has shown that a superstable group has a finite series whose factors are either abelian or simple groups. Our main result is:

Theorem 1.1. *Let G be an infinite supersimple group. Then there is a finite chain of definable subgroups $1 = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$ such that every quotient H_{i+1}/H_i is infinite and either virtually FC or virtually simple modulo a finite FC-centre.*

Consequently, Baudisch's result extends to supersimple groups. His arguments however do not adapt to the supersimple context, mainly for two reasons: first, the proof of [Bau] is based on a transfinite induction on the U-rank of G, the induction basis being Berline and Lascar's result that a superstable group of U-rank 1 is virtually abelian; second, [Bau] makes a strong use of the connected component of a superstable group; in a supersimple group, there is also a notion of connected component, but it heavily depends on the specific parameter set over which it is defined. So we have to work without the use of Berline and Lascar's result and without connected components.

For the development of a suitable version of the result by Berline and Lascar, our basic idea is provided by [Wag00, Remark 5.4.11]: a supersimple group of U-rank 1 either has an infinite abelian subgroup or is virtually simple modulo a finite FC-centre; this makes a proof by transfinite induction still possible. To avoid the use of connected components, we study just-infinite supersimple groups. These turn out to be either virtually FC or virtually simple.

Other important tools are Lascar's additive properties of the SU-rank, Wagner's version of Zilber's Indecomposability Theorem for supersimple groups and the observation that the FC-centre of a supersimple group is defined by a first order formula. It should be mentioned that these methods mainly come from Berline and Lascar's investigation of superstable groups in [BL], later extended by Wagner to supersimple groups [Wag00].

The main theorem has several consequences. We outline three of them here. Sela has recently shown that the first order theory of a torsion-free hyperbolic group is stable, hence simple. Using a theorem of Delzant [Del], we derive:

Corollary 1.2. *A non-elementary hyperbolic group is not supersimple.*

Jaligot, Elwes, Macpherson and Ryten have proved in [EJMR] that the soluble radical of a supersimple group of finite SU-rank satisfying an additional technical assumption is soluble and definable.