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Ordinary and graded cocharacter of the Jordan algebra of 2 × 2 upper triangular matrices

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ABSTRACT

Let F be a field of characteristic zero and $UJ_2(F)$ be the Jordan algebra of 2×2 upper triangular matrices over F. In this paper we give a complete description of the space of multilinear graded and ordinary identities in the language of Young diagrams through the representation theory of a Young subgroup of S_n. For every $\mathbb{Z}_2$-grading of $UJ_2(F)$ we compute the multiplicities in the graded cocharacter sequence and furthermore we compute the ordinary cocharacter.

1. Introduction

Let A be an associative algebra over a field F of characteristic 0, and denote by $Id(A)$ its T-ideal. Since char $F = 0$ it suffices to study only the multilinear polynomial identities of A. Let P_n be the vector space of the multilinear polynomials in $x_1, \ldots, x_n$ in the free

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associative algebra $F\langle X\rangle$. We assume that the set X of the free generators is countable and infinite. Thus in order to study the identities of A one studies the intersections $P_n \cap Id(A)$, $n \geqslant 1$. But for practical purposes these intersections are not suitable since they tend to become very large as $n \to \infty$. Therefore one is led to study the quotients $P_n(A) = P_n/(P_n \cap T(A))$. The dimension $c_n = c_n(A) = \dim P_n(A)$ is called the n-th *codimension* of A; the sequence of codimensions for a given algebra is one of the most important characteristics of the identities of A. In [6,8] Giambruno and Zaicev proved that the sequence $(c_n(A))^{1/n}$ converges, and its limit is always an integer, called the PI-*exponent* of A.

One may define and study analogous concepts for large classes of nonassociative algebras as well.

The algebras $UT_n(F)$ of upper triangular matrices are one of the first classes of algebras to have their identities completely described. They are crucial in classifying the subvarieties of the variety of algebras generated by the matrix algebra of order 2. Here we mention that concrete bases of identities for an algebra are known in few cases. The identities of the matrix algebra $M_2(F)$ are known over any field as long as $\operatorname{char} F \neq 2$, see [18,15,10]; bases for the Grassmann algebra E are also known over any field, see [14,13,19,20]. In [17] the identities of $E \otimes E$ when $\operatorname{char} F = 0$ were described. The identities of $UT_n(F)$ are also known. Concerning Lie algebras, a basis of the identities of $sl_2(F)$ is known, see [18,21]. The identities of the Lie algebra $UT_n(F)$ are easy to describe. In the case of Jordan algebras, the only nontrivial case where the identities are known are those of the algebras B_n and B of a nondegenerate symmetric bilinear form. These results are due to Vasilovsky [22]. Nowadays, in [16] the author computed a basis of the identities of B_n equipped with a degenerate symmetric bilinear form of rank $n-1$. Recall that earlier Iltyakov [9] had developed methods to study the identities in these algebras and had proved that the variety generated by B_n is Spechtian. Apart from the results mentioned above one does not know the concrete form of the identities satisfied by given algebra.

Gradings on algebras and the corresponding graded identities have become an area of extensive study. We refer the interested reader to the survey [2] for further reading and reference (see also [11]) concerning gradings and graded identities.

In contrast with the associative case, graded identities for Lie and Jordan algebras have seldom been studied. Among the few known results we mention [11]. In the first of these papers, the graded identities for the Lie algebra $sl_2(F)$ were described, under every possible grading. The second paper dealt with the graded identities of the Jordan algebra of the symmetric matrices of order two.

Another important tool in the study of P.I. algebras, is the character of $P_n(A)$ according to the representation theory of the symmetric group S_n. If the P.I. algebra is ordinary or G-graded this character is called n-th cocharacter and n-th graded cocharacter respectively. In this paper we study the sequence of the n-th cocharacter and of the n-th graded cocharacter for the Jordan algebra $UJ_2(F)$ of the upper triangular matrices of order 2 over an infinite field of characteristic zero, finding explicitly the multiplicities. In [12] the authors studied all gradings on UJ_2 by the group $\mathbb{Z}_2$ obtaining the bases of