Applied Mathematics and Computation 220 (2013) 405–413

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Distribution of some graph invariants over hierarchical product of graphs

M. Tavakoli [a,*], F. Rahbarnia [a], A.R. Ashrafi [b]

[a] Department of Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Islamic Republic of Iran
[b] Department of Mathematics, Faculty of Mathematics, Statistics and Computer Science, University of Kashan, Kashan 87317-51167, Islamic Republic of Iran

ARTICLE INFO

Keywords:
Hierarchical product
Median graph
Eccentric distance sum
Metric dimension
Edge revised Szeged index

ABSTRACT

The hierarchical product of graphs was introduced very recently by L. Barriere et al. in [On the spectra of hypertrees, Linear Algebra Appl. 428 (2008) 1499–1510], and some of its main properties were studied. In this paper, some new properties of this new graph product are investigated. We prove that $G_n \sqcap \ldots \sqcap G_1$ is median graph if and only if $G_1, G_2, \ldots, G_n$ are median. An exact formula for metric dimension of $G_n \sqcap \ldots \sqcap G_1$, as well as formulas for the eccentric distance sum and edge revised Szeged of hierarchical product of graphs are presented. Some applications of our results are also presented.

1. Introduction

In this section we recall some definitions that will be used in the paper. The **Graph invariants** are parameters that are preserved under graph isomorphisms. However, they are not usually preserved under graph homomorphisms. A **topological index** is a graph invariant applicable in chemistry.

Suppose G is a graph with vertex and edge sets of $V(G)$ and $E(G)$, respectively. If $x, y \in V(G)$ then the **distance** $d_G(x, y)$ (or d (x,y) for short) between x and y is defined as the length of a minimum path connecting x and y. The **Wiener index** of G, $W(G)$, is defined as the summation of distances between all pairs of vertices in G. In other words, The Wiener index index of a graph G is defined as $\mathrm{W}(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v)$ [27]. A topological index is called distance-based if it can be defined by distance function $d(-,-)$. It is worthy to mention here that Wiener did not consider the distance function $d(-,-)$ in his seminal paper. Hosoya [15], presented a new simple formula for the Wiener index by using distance function. We encourage the readers to consult [8,9] for more information on Wiener index.

The eccentricity $\varepsilon_G(u)$ is defined as the largest distance between u and other vertices of G. We will omit the subscript G when the graph is clear from the context. The maximum eccentricity over all vertices of G is called the diameter of G and denoted by $D(G)$ and the minimum eccentricity among the vertices of G is called radius of G and denoted by $r(G)$. The eccentric connectivity index of a graph G is defined as $\xi(G) = \sum_{u \in V(G)} deg_G(u)\varepsilon_G(u)$ [23]. We encourage the interested readers to consult papers [3,4] for applications of this graph invariant in chemistry and [30,31,16,20] for its mathematical properties.

For a given vertex $u \in V(G)$ we define its **distance sum** $D_G(u)$ as $D_G(u) = \sum_{v \in V(G)} d_G(u, v)$. The **eccentric distance sum** of the graph G is obtained by summing over all vertices of G their distance sums multiplied by their respective eccentricities [10]. Hence, $\xi^{SD}(G) = \sum_{u \in V(G)} D_G(u)\varepsilon_G(u)$. The concept of eccentricity also gives rise to a number of other topological invariants. For example, the **total eccentricity** $\zeta(G)$ of a graph G that is defined as $\zeta(G) = \sum_{u \in V(G)} \varepsilon_G(u)$.

The interval $I(u, v)$ between two vertices u and v of a graph G is the set of vertices on shortest paths between u and v. Note that $I(u, v)$ contains u and v. A median for a triple of vertices u, v, w of a graph G is a vertex z that lies on a shortest u, v-path,

* Corresponding author.
E-mail addresses: Mostafa.Tavakoli@stu-mail.um.ac.ir, m.tavakoly@ut.ac.ir (M. Tavakoli).

0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2013.06.009

on a shortest u, w-path, and on a shortest v, w-path. Note that z can be one of the vertices u, v, w. Alternatively, the medians of u, v, w can be defined as the vertices in $I(u,v) \cap I(u,w) \cap I(v,w)$. A **median** graph is a simple graph that every triple of vertices has a unique median, namely if

$$|I(u,v) \cap I(u,w) \cap I(v,w)| = 1$$

for every triple $u, v, w \in V(G)$ [13].

Let $G = (V, E)$ be a simple graph of order $n = |V|$. Given $u, v \in V, u \sim v$ means that u and v are adjacent vertices. Given a set of vertices $S = \{v_1, v_2, \ldots, v_k\}$ of a connected graph G, the **metric representation** of a vertex $v \in V$ with respect to S is the vector $r(v|S) = (d_G(v,v_1), d_G(v,v_2), \ldots, d_G(v,v_k))$. We say that S is a **resolving set** for G if for every pair of distinct vertices $u, v \in V, r(u|S) \neq r(v|S)$. The **metric dimension** of G is the minimum cardinality of any resolving set for G, and it is denoted by $dim(G)$.

For an edge $e = ab$ of G, let $n_a(e)$ be the number of vertices closer to a than to b. In other words, $n_a^G(e) = |\{u \in V(G) | d(u,a) < d(u,b)\}|$. In addition, let $n_0^G(e)$ be the number of vertices with equal distances to a and b, i. e., $n_0^G(ab) = |\{u \in V(G) | d(u,a) = d(u,b)\}|$. We also denote the number of edges of G whose distance to the vertex a is smaller than the distance to the vertex b by $m_a^G(e)$. Moreover, the number of edges with equal distances to a and b is denoted by $m_0^G(ab)$. The Szeged, revised Szeged, edge revised Szeged, vertex-edge Szeged, modified vertex-edge Szeged, vertex Padmakar–Ivan and edge Padmakar–Ivan indices of the graph G is defined as:

$$\begin{aligned}
\mathrm{Sz}_v(G) &= \sum_{e=uv\in E(G)} n_u^G(e) n_v^G(e),\\
\mathrm{Sz}_v^*(G) &= \sum_{e=uv\in E(G)} \left(n_u^G(e) + \frac{n_0^G(e)}{2}\right)\left(n_v^G(e) + \frac{n_0^G(e)}{2}\right)\\
\mathrm{Sz}_e^*(G) &= \sum_{e=uv\in E(G)} \left(m_u^G(e) + \frac{m_0^G(e)}{2}\right)\left(m_v^G(e) + \frac{m_0^G(e)}{2}\right)\\
\mathrm{Sz}_{ev}(G) &= \frac{1}{2}\sum_{e=uv\in E(G)} (m_u^G(e) n_v^G(e) + m_v^G(e) n_u^G(e))\\
\mathrm{Sz}_{ev}^*(G) &= \frac{1}{2}\sum_{e=uv\in E(G)} (m_u^G(e) n_u^G(e) + m_v^G(e) n_v^G(e))\\
\mathrm{PI}_v(G) &= \sum_{e=uv\in E(G)} (n_u^G(e) + n_v^G(e))\\
\mathrm{PI}_e(G) &= \sum_{e=uv\in E(G)} (m_u^G(e) + m_v^G(e)).
\end{aligned}$$

see for details [11,12,21,22,19].

Let $G_i = (V_i, E_i)$ be N graphs with each vertex set $V_i, 1 \leqslant i \leqslant N$, having a distinguished or root vertex, labeled 0. The **hierarchical product** $H = G_N \sqcap \ldots \sqcap G_2 \sqcap G_1$ is the graph with vertices the $N-$tuples $x_N \ldots x_3 x_2 x_1, x_i \in V_i$, and edges defined by the adjacencies:

$$x_N \ldots x_3 x_2 x_1 \sim \begin{cases} x_N \ldots x_3 x_2 y_1 & \text{if } y_1 \sim x_1 \quad \text{in } G_1,\\ x_N \ldots x_3 y_2 x_1 & \text{if } y_2 \sim x_2 \quad \text{in } G_2 \text{ and } x_1 = 0,\\ x_N \ldots y_3 x_2 x_1 & \text{if } y_3 \sim x_3 \quad \text{in } G_3 \text{ and } x_1 = x_2 = 0,\\ \vdots & \vdots \\ y_N \ldots x_3 x_2 x_1 & \text{if } y_N \sim x_N \quad \text{in } G_N \text{ and } x_1 = x_2 = \cdots = x_{N-1} = 0. \end{cases}$$

As an example, Fig. 1 shows the hierarchical product $C_4 \sqcap C_4 \sqcap C_4$. We encourage the reader to consult [5,6,24] for the mathematical properties of hierarchical product of graphs.

Throughout this paper we consider graphs means simple connected graphs, connected graphs without loops and multiple edges. We denote the path and cycle graphs of order n by P_n and C_n, respectively. A graph G is called nontrivial if $|V(G)| > 1$. A graph G is said to be (vertex) **distance-balanced**, if $n_a^G(e) = n_b^G(e)$, for each edge $e = ab \in E(G)$, see [1,17,25] for details. These graphs first studied by Handa [14] who considered distance-balanced partial cubes. In [18], Jerebič et al. studied distance-balanced graphs in the framework of various kinds of graph products. After that, in [26], the present authors introduced the concept of **edge distance-balanced graphs**. Such a graph G has this property that $m_a^G(e) = m_b^G(e)$ holds for each edge $e = ab \in E(G)$.

2. Main results

In this section our main results are presented. We start by medianness of the hierarchical product product of graphs.