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Stability analysis of the inverse Lax–Wendroff boundary treatment for high order upwind-biased finite difference schemes

Tingting Li [a], Chi-Wang Shu [b,*], Mengping Zhang [a]

[a] School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, PR China

[b] Division of Applied Mathematics, Brown University, Providence, RI 02912, USA

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ABSTRACT

In this paper, we consider linear stability issues for one-dimensional hyperbolic conservation laws using a class of conservative high order upwind-biased finite difference schemes, which is a prototype for the weighted essentially non-oscillatory (WENO) schemes, for initial–boundary value problems (IBVP). The inflow boundary is treated by the so-called inverse Lax–Wendroff (ILW) or simplified inverse Lax–Wendroff (SILW) procedure, and the outflow boundary is treated by the classical high order extrapolation. A third order total variation diminishing (TVD) Runge–Kutta time discretization is used in the fully discrete case. Both GKS (Gustafsson, Kreiss and Sundström) and eigenvalue analyses are performed for both semi-discrete and fully discrete schemes. The two different analysis techniques yield consistent results. Numerical tests are performed to demonstrate the stability results predicted by the analysis.

1. Introduction

When a high order finite difference scheme with wide stencil is used to solve hyperbolic conservation laws, the inner schemes cannot be used near the boundary. Special treatments near the boundaries are needed in order to maintain accuracy and stability. There exist two difficulties when imposing numerical boundary conditions. Firstly, the points used in these schemes which lie outside the computational domain, namely the "ghost points", should be evaluated properly. Secondly, the grid points may not coincide with the physical boundary exactly. For hyperbolic conservation laws, classical Lagrangian extrapolation to evaluate ghost point values near the outflow boundary usually leads to stable approximations. However, it is a challenge to obtain stable and accurate numerical boundary conditions near the inflow boundary. This is especially the case when the physical boundary does not coincide with but is very close to the first grid point, which is referred to as the "cut-cell" problem in the literature, see e.g. [1]. The inverse Lax–Wendroff (ILW) procedure, first introduced in [2], can overcome this difficulty. The simplified ILW (SILW) procedure, which is an extension of the ILW procedure and can save in algorithm complexity and computational cost, is introduced in [3]. For earlier related work, see [4–8].

Stability of the numerical schemes for initial–boundary value problems (IBVP) on finite domain can be established by the normal mode analysis, which is based on the Laplace transform. General stability analysis based on this technique is the famous Gustafsson, Kreiss and Sundström (GKS) theory [9]. In [9], stability of fully discrete finite difference schemes is

* Corresponding author.
E-mail addresses: ltt1120@mail.ustc.edu.cn (T. Li), shu@dam.brown.edu (C.-W. Shu), mpzhang@ustc.edu.cn (M. Zhang).

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analyzed. In [10], stability analysis is performed for the semi-discrete cases. For the ILW and SILW procedures when high order central compact spatial operations are used, such GKS analysis has been performed in [11]. However, the GKS analysis may lead to high algebraic complexity in the case of very high order accuracy. An alternative technique, by visualizing the eigenvalue spectrum of the discretization operators, has also been used in [11] to analyze stability. It has been observed in [11] that, when both techniques are used, they produce consistent stability conclusions. In this paper, we are interested in studying stability of semi-discrete and fully discrete upwind-biased high order finite difference schemes, which serve as a prototype for the weighted essentially non-oscillatory (WENO) schemes [12–14]. ILW and SILW procedures will be used near the inflow boundary and classical Lagrangian extrapolation will be used near the outflow boundary. Both the GKS analysis and the eigenvalue analysis will be performed.

This paper is organized as follows. In Section 2, we review high order upwind-biased finite difference methods as the inner schemes, and the third order total variation diminishing (TVD) Runge–Kutta time discretization method used in the full discretization. The ILW procedure, the SILW procedure and classical extrapolation are introduced in detail also in this section. Stability analysis is performed in Section 3 by using the GKS theory and the eigenvalue spectrum method, first for the semi-discrete case and then for the fully discrete case. Numerical tests are provided in Section 4 to demonstrate the results of the analysis. Concluding remarks are given in Section 5.

2. Scheme formulation

In this section, we review high order upwind-biased finite difference methods as the inner schemes, and the third order total variation diminishing (TVD) Runge–Kutta time discretization method used in the full discretization. We also introduce the ILW procedure, the SILW procedure and classical extrapolation used in the inflow and outflow boundary treatments.

2.1. High order upwind-biased finite difference schemes

Consider the one-dimensional scalar conservation law

$$\begin{cases} u_t + f(u)_x = 0, & x \in [a, b],\ t \geq 0 \\ u(a, t) = g(t), & t \geq 0 \\ u(x, 0) = u_0(x), & x \in [a, b]. \end{cases} \tag{2.1}$$

Assume that $f'(u(a, t)) > 0$ and $f'(u(b, t)) > 0$ for $t > 0$. This assumption guarantees that the left boundary $x = a$ is an inflow boundary where a boundary condition is needed, and the right boundary $x = b$ is an outflow boundary where no boundary condition can be prescribed.

The interval (a, b) is discretized by a uniform mesh as

$$a + C_a\,\Delta x = x_0 < x_1 < x_2 < \cdots < x_N = b - C_b\,\Delta x \tag{2.2}$$

where $C_a \in [0, 1)$ and $C_b \in [0, 1)$. $\{x_j = a + (C_a + j)\,\Delta x,\ j = 0, 1, 2, \ldots, N\}$ are the grid points. The first and the last grid points are not necessarily aligned with the boundary, and we choose this kind of discretization on purpose.

The general semi-discrete conservative finite difference scheme approximating (2.1), based on point values and numerical fluxes, is of the form:

$$\frac{du_j}{dt} = -\frac{1}{\Delta x}\left(\hat{f}_{j+\frac{1}{2}}(t) - \hat{f}_{j-\frac{1}{2}}(t)\right) \tag{2.3}$$

where the numerical flux $\hat{f}_{j+\frac{1}{2}}$ is defined as a linear combination of $f(u(x, t))$ in the neighborhood of x_j such that the right hand of (2.3) approximates $-f(u)_x$ at $x = x_j$ to the desired order of accuracy.

For convenience, the semi-discrete approximation (2.3) can be written as

$$\frac{du_j}{dt} = -\frac{1}{\Delta x}\mathcal{D}_{j,\hat{k},\hat{m}} \cdot f_j \equiv -\frac{1}{\Delta x}\sum_{l=0}^{\hat{m}} d_{\hat{k},l}\, f_{j-\hat{k}+l} \tag{2.4}$$

where $j - \hat{k}$ denotes the left most point of the derivative stencil having $\hat{m} + 1$ points. Both $\hat{k}$ and $\hat{m}$ depend on the order of the scheme.

Schemes considered in this paper are listed below.

- Third order scheme

$$\frac{du_j}{dt} = -\frac{1}{\Delta x}\left(\frac{1}{6}f_{j-2} - f_{j-1} + \frac{1}{2}f_j + \frac{1}{3}f_{j+1}\right).$$

- Fifth order scheme

$$\frac{du_j}{dt} = -\frac{1}{\Delta x}\left(-\frac{1}{30}f_{j-3} + \frac{1}{4}f_{j-2} - f_{j-1} + \frac{1}{3}f_j + \frac{1}{2}f_{j+1} - \frac{1}{20}f_{j+2}\right).$$