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Foliation on the moduli space of extrinsic circular trajectories on a complex hyperbolic space

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ABSTRACT

We show that every circle on a complex hyperbolic space except unbounded circles of complex torsion ± 1 which have two distinct points at infinity can be regarded as a trajectory for some Sasakian magnetic field on some totally η-umbilic real hypersurface. Our study explains why the lamination on the moduli space of circles on a complex hyperbolic space has singularities only on the leaf of circles of complex torsion ± 1.

1. Introduction

A smooth curve γ parameterized by its arclength is said to be a *circle* if it satisfies the differential equation $\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + k^2\dot\gamma = 0$ with some nonnegative constant k. Under the assumption that γ is of unit speed, this equation is equivalent to the system of equations $\nabla_{\dot\gamma}\dot\gamma = kY$, $\nabla_{\dot\gamma}Y = -k\dot\gamma$ with a unit vector field Y along γ. The nonnegative constant k is called the *geodesic curvature* and the frame field $\{\dot\gamma, Y\}$ is called the Frenet frame of γ. Since circles of null geodesic curvature are geodesics, we may say that circles are simplest curves next to geodesics. Still, as we need information on both their velocities and accelerations, it is not so easy to investigate Riemannian manifolds by making use of them. We have a few results in such a direction (see [15,13] and so on).

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In this paper we study circles on a complex hyperbolic space from the viewpoint of magnetic fields on its real hypersurfaces. A *magnetic field* is a closed 2-form on a Riemannian manifold. Typical examples of magnetic fields are *Kähler magnetic fields* on a Kähler manifold and *Sasakian magnetic fields* on its real hypersurfaces [1,11]. They are constant multiples of the Kähler form and those of the canonical form associated with the almost contact metric structure induced by the ambient Kähler structure (see Section 3 for more detail). A motion of a charged particle of unit speed under the action of a magnetic field is said to be a *trajectory*. Since a motion of a charged particle without the action of magnetic fields is a geodesic, trajectories for magnetic fields are generalizations of geodesics. If we recall circles on a Euclidean space $\mathbb{R}^n$, every circle is the "extrinsic shape" of a geodesic on some geodesic sphere S^{n-1} in $\mathbb{R}^n$. In view of this fact, we explain circles on a complex hyperbolic space as extrinsic shapes of trajectories on its totally η-umbilic real hypersurfaces, which are geodesic spheres, horospheres and tubes around totally geodesic complex hypersurfaces.

The reason why we stick to such an explanation is that the behavior of the length function of circles on a nonflat complex space form is not consistent with natural parameters of congruence classes of circles under the action of isometries. For circles of positive geodesic curvature on a Kähler manifold, their congruence classes under the action of isometries are parameterized by their geodesic curvatures and complex torsions. Here, complex torsions of circles are defined by their Frenet frame and by complex structure (see Section 2). On the moduli space, the set of all congruence classes of circles, we have a lamination associated with lengths of circles [3]. Since this structure has singularities only on the leaf of circles whose velocities and accelerations form complex lines, we are interested in giving an explanation of this phenomena. We should note that on the moduli space of circles and ordinary helices on a real hyperbolic space we have corresponding foliation [5]. From the viewpoint of extrinsic shapes, circles whose velocities and accelerations form complex lines and other circles on a complex hyperbolic space are quite different. The author considers that it is better to take the former to be trajectories for Kähler magnetic fields and the latter to be extrinsic shapes of trajectories for Sasakian magnetic fields on totally η-umbilic real hypersurfaces. Along our study we show that we have a natural foliation on the moduli space of these trajectories whose extrinsic shapes are circles.

2. Lamination on the moduli space of circles on a complex hyperbolic space

In order to make clear our point of view, we shall start by recalling properties of circles on a complex hyperbolic space. A smooth curve γ parameterized by its arclength is said to be *closed* if there is a positive t_p satisfying $\gamma(t+t_p)=\gamma(t)$ for all t. The minimum positive t_p satisfying this property is called the *length* of γ and is denoted by $\mathrm{length}(\gamma)$. For an open curve γ, a curve which is not closed, we set $\mathrm{length}(\gamma)=\infty$. To investigate a Riemannian manifold by make use of properties of curves on this manifold, it is a basic idea to study the distribution of their lengths. We say two curves γ_1, γ_2 on a Riemannian manifold M parameterized by their arclength to be *congruent* to each other if there exist an isometry φ of M and a constant t_c satisfying $\gamma_2(t)=\varphi\circ\gamma_1(t+t_c)$ for all t. For example, circles on a real hyperbolic space $\mathbb{R}H^n$ are congruent to each other if and only if they have the same geodesic curvature. Hence the moduli space, which is the set of all congruence classes, of circles on $\mathbb{R}H^n$ is set theoretically identified with a half line $(0,\infty)$. When $\mathbb{R}H^n$ is of constant sectional curvature c, then a circle is closed if and only if its geodesic curvature is greater than $\sqrt{|c|}$, and in this case its length is $2\pi/\sqrt{k^2+c}$. Thus if we consider the length function on the moduli space $(0,\infty)$ of circles on $\mathbb{R}H^n$, it is smooth with respect to the ordinary differentiable structure on $(0,\infty)\subset\mathbb{R}$.

For circles on a Kähler manifold M with complex structure J, we have another invariant. Given a circle γ with Frenet frame $\{\dot\gamma, Y\}$ on M, we set $\tau_{12}=\langle\dot\gamma, JY\rangle$, and call it its *complex torsion*. Clearly it is constant along γ and satisfies $|\tau_{12}|\le 1$. On a complex hyperbolic space $\mathbb{C}H^n$, two circles are congruent to each other under the action of holomorphic isometries if they have the same geodesic curvature and the same complex torsion (see [14]). Thus the moduli space of circles on $\mathbb{C}H^n$ under the action of holomorphic isometries is