Annals of Pure and Applied Logic 164 (2013) 837–854

Contents lists available at SciVerse ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Automatic models of first order theories

Pavel Semukhin [a,∗,1], Frank Stephan [b,2]

[a] Department of Computer Science, University of Regina, Canada
[b] Department of Mathematics and Department of Computer Science, National University of Singapore, Singapore 119076, Republic of Singapore

ARTICLE INFO

Article history:
Received 13 July 2012
Received in revised form 18 March 2013
Accepted 21 March 2013
Available online 29 March 2013

MSC:
03C50
03C30
03D05
68Q45

Keywords:
Automatic structures
Model theory

ABSTRACT

Khoussainov and Nerode (2008) [14] posed various open questions on model-theoretic properties of automatic structures. In this work we answer some of these questions by showing the following results: (1) There is an uncountably categorical but not countably categorical theory for which only the prime model is automatic; (2) There are complete theories with exactly $3, 4, 5, \ldots$ countable models, respectively, and every countable model is automatic; (3) There is a complete theory for which exactly 2 models have an automatic presentation; (4) If $LOGSPACE = P$ then there is an uncountably categorical but not countably categorical theory for which the prime model does not have an automatic presentation but all the other countable models are automatic; (5) There is a complete theory with countably many countable models for which the saturated model has an automatic presentation but the prime model does not have one.

1. Introduction

This paper is devoted to the study of automatic structures from the model-theoretic point of view. Automatic structures are the algebraic structures whose functions and relations can be recognised by finite automata. Historically, this notion was introduced in the work of Hodgson [12], and later in Khoussainov and Nerode [13] and Blumensath and Grädel [2].

Automatic structures are famous in theoretical computer science because of their decidability properties. Namely, the *model checking problem* for automatic structures is decidable. In other words, there is an algorithm that, given an automatic structure $\mathcal{A}$, a first order formula $\varphi(\bar{x})$ and a tuple $\bar{a}$ of elements from $\mathcal{A}$, it decides whether $\mathcal{A} \models \varphi(\bar{a})$. In particular, automatic structures are decidable structures (a structure $\mathcal{A}$ is *decidable* if there is an algorithm that decides whether $\mathcal{A} \models \varphi(\bar{a})$ holds, for any first order formula $\varphi(\bar{x})$ and any tuple $\bar{a} \in \mathcal{A}$). This, in turn, implies that the first order theory of any automatic structure is decidable. One can use this property to prove the decidability of first order theories for many mathematical structures, for example, the Presburger arithmetic $(\mathbb{N}, +)$.

A lot of work has been devoted to study the question as to which structures have automatic presentations [3,5,15,20,21,23]. In some cases we have an elegant characterisation of automatic structures in a given class, for instance [5,15]:

- An ordinal α is automatic if and only if $\alpha < \omega^\omega$.

∗ Corresponding author.
E-mail addresses: pavel@semukhin.name (P. Semukhin), fstephan@comp.nus.edu.sg (F. Stephan).

1 P. Semukhin was supported by NUS grant R146-000-114-112 and later by a Post-doctoral Fellowship from the Pacific Institute for the Mathematical Sciences and in part by IMS programme "Automata Theory and Applications" during his visit at NUS.
2 Supported in part by NUS grants R146-000-114-112 and R252-000-420-112.

0168-0072/$ – see front matter
http://dx.doi.org/10.1016/j.apal.2013.03.001

- The additive semigroup of an ordinal ω^α is automatic if and only if $\alpha < \omega$.
- A Boolean algebra $\mathcal{B}$ is automatic if and only if $\mathcal{B}$ is either finite or isomorphic to B_ω^n, where B_ω is the algebra of all finite and co-finite subsets of the natural numbers.

On the other hand, it is still an open problem to describe automatic Abelian groups and automatic linear orders. Another important question that attracted attention is how difficult are the isomorphism problems for various classes of structures [15,18,19].

Recently, Khoussainov and Nerode [14] linked automatic structures to model theory and posed a list of important questions arising from this connection. Some of those questions are answered in the present work. The interested reader is referred to Khoussainov and Rubin [17,22] for general surveys on automatic structures.

To describe in more detail what it means for a structure to be automatic, we need a notion of convolution. A *convolution* of two strings u and v in an alphabet Σ is a string $Conv(u, v)$ in alphabet $(\Sigma \cup \{\square\})^2$ which is obtained by putting u above v. If the strings have different length, then we use a new padding symbol $\square$ to fill in the shorter string. For example, if $u = ab$ and $v = bbaa$, then

$$Conv(ab, bbaa) = \begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} b \\ b \end{pmatrix}\begin{pmatrix} \square \\ a \end{pmatrix}\begin{pmatrix} \square \\ a \end{pmatrix}.$$

Similarly, one can define a convolution of several finite strings.

An n-ary relation R on finite strings is called *automatic* if the set of convolutions of all tuples from R is recognizable by a finite automaton. A function is called *automatic* if its graph is automatic. Intuitively, an automatic function can be thought of as a one whose result can be computed “locally”, using a fixed finite amount of external memory. A typical example of an automatic function is the addition operation on natural numbers in binary presentation. In this case, an automaton computes the addition bitwise, remembering only one carry bit when necessary.

An algebraic structure $\mathcal{A}$ is called *automatic* if its domain, functions and relations are automatic. A structure is said to have *an automatic presentation* if it is isomorphic to an automatic structure. Typical examples of automatic structures are [13,15,21]:

- the ordering of the natural numbers $(\omega, \leqslant)$ and the ordering of the rationals $(\mathbb{Q}, \leqslant)$;
- finitely generated Abelian groups, in particular, the ordered group $(\mathbb{Z}, +, \leqslant)$;
- Prüfer groups $\mathbb{Z}(p^\infty)$ for every prime p;
- countably dimensional vector spaces over finite fields;
- the Boolean algebra of all finite and co-finite subsets of $\mathbb{N}$.

On the other hand, the following structures do not have automatic presentations [5,15]:

- the natural numbers with multiplication $(\mathbb{N}, \times)$;
- the free group over more than one generator;
- the random graph;
- the atomless Boolean algebra.

There was a long standing open question whether the group of rationals under addition $(\mathbb{Q}, +)$ is automatic. This question was recently solved by Tsankov [23] who showed that $(\mathbb{Q}, +)$ does not have an automatic presentation.

The topic of our work was inspired by computable model theory. Computable structures are generalisations of automatic structures in the sense that their operations are computable by Turing machines, rather than by finite automata. One of the general questions studied in computable model theory can be stated as follows: given a complete first order theory T, which models of T have computable presentations? In particular, when the prime or the saturated model of T is computable?

The latter question is especially interesting in the case of uncountably categorical theories (recall that T is called *uncountably categorical* if any two models of T of cardinality $\aleph_1$ are isomorphic). In [9,10,16], examples of such theories were constructed with the following properties:

(1) only the prime model of a theory is computable;
(2) only the saturated model of a theory is computable;
(3) all models except the prime one are computable;
(4) all models except the saturated one are computable.

For our purpose, we will reformulate the above mentioned question as follows: *given a complete first order theory T, describe the automatic models of T*. This general question can be divided into a number of more specific subquestions. In our paper we consider the following questions, which are stated as Question 3.2 and Question 3.3 by Khoussainov and Nerode [14].

Question 3.2. Let $n \geqslant 1$ be a natural number. Does there exist a theory with exactly n automatic models up to isomorphism? (When $n = 1$, the theory should not be $\aleph_0$-categorical.)