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On compactifications and the topological dynamics of definable groups

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A B S T R A C T

For G a group definable in some structure M, we define notions of "definable" compactification of G and "definable" action of G on a compact space X (definable G-flow), where the latter is under a definability of types assumption on M. We describe the universal definable compactification of G as $G^*/(G^*)^{00}_M$ and the universal definable G-ambit as the type space $S_G(M)$. We also point out the existence and uniqueness of "universal minimal definable G-flows", and discuss issues of amenability and extreme amenability in this definable category, with a characterization of the latter. For the sake of completeness we also describe the universal (Bohr) compactification and universal G-ambit in model-theoretic terms, when G is a topological group (although it is essentially well-known).

1. Introduction and preliminaries

Given a topological (Hausdorff) group G, a compactification of G is a pair (H, f) where H is a compact topological group and $f : G \to H$ a continuous homomorphism (not necessarily injective) with dense image. There is a *universal* such compactification, called the *Bohr compactification*. Let us note immediately that a compactification of the topological group G is a special case of continuous action of G on a compact space X, where X has a distinguished point x_0 with dense orbit under G (a so-called G-ambit). Again

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there is a universal G-ambit. There is reasonably comprehensive account of abstract topological dynamics in [1].

On the other hand, given a group G-definable in a structure M, a saturated elementary extension M^* of M, G^* the interpretation of the formulas defining G in M^* and a type-definable over M, normal subgroup N of G^* of index at most $2^{|M|}$, G^*/N, equipped with the so-called *logic topology* is a compact Hausdorff group, and the identity embedding of G in G^* induces a homomorphism from G into G^*/N with dense image. There is a smallest such N which is called $(G^*)^{00}_M$, a kind of *connected component* of G^*. More generally we have the action of G on the space $S_G(M)$ of complete types over M concentrating on G, where $S_G(M)$ is the Stone space of the Boolean algebra of definable subsets of G. Moreover there is a canonical dense orbit: $G \cdot \mathrm{tp}(1_G/M)$.

Here we will relate these two theories, in the context of two categories: first the classical case of topological groups, and secondly the new case of definable groups. The case of compactifications of topological groups was explicated by Robinson and Hirschfeld [14,5] in the language of nonstandard analysis. For the more general case of G-flows, the explication of the universal G-ambit via the *Samuel compactification* of G [15,18], with respect to the right uniformity on G, is basically equivalent to the model-theoretic account that we give below. As far as definable groups are concerned, we give appropriate definitions (definable compactification, definable G-flow), obtaining in a sense a theory of *tame* topological dynamics, although we make a *definability of types* assumption on the model concerned, in the case of group actions. A free group $\mathbb{F}_n$ ($n \geqslant 2$), considered as a first order structure $(\mathbb{F}_n, \cdot)$, will be extremely amenable as a definable group, although considered as a discrete topological group, it will not be such.

A big influence on this paper is work of Newelski (for example [8,9]) on trying to use the machinery of topological dynamics to extend stable group theory to tame unstable contexts. The present paper is quite soft, aiming partly at putting Newelski's ideas into a formal framework, *definable, or tame, topological dynamics*. On the other hand the treatment of *fsg* groups in [12] and the case analysis of $\mathrm{SL}(2,\mathbb{R})$ in [4] from this point of view, are rather harder.

In the remainder of this introduction, we recall some very basic model-theoretic notions and constructions. A good reference for the kind of model theory in the current paper is [13] as well as [6]. Whenever we take about a topological space we assume Hausdorffness.

We fix a complete theory T in language L, a model M of T and a very saturated elementary extension M^* of M, for example κ-saturated of cardinality κ where $\kappa > 2^{|M|+|L|}$. For X a definable set in M^*, definable over M (or even a set definable in M), $S_X(M)$ denotes the Stone space of complete types over M which contain (the formula defining) X. By a type-definable over M set in M^* we mean the common solution set in M^* of a collection of formulas over M, equivalently an intersection of sets, definable in M^* over M. Sometimes L_M denotes the language L expanded by constants for elements of M.

We first recall the logic topology on bounded hyperdefinable sets (in a saturated model).

Definition 1.1. Let X be a definable set in the structure M^*, definable with parameters from M.

(i) Suppose E is a type-definable over M equivalence relation with a *bounded* number of classes, that is $< \kappa$ (equivalently $\leqslant 2^{|M|+|L|}$) many, and $\pi : X \to X/E$ the canonical surjection. The *logic topology* on X/E is defined as follows: $Z \subseteq X/E$ is closed if $\pi^{-1}(Z) \subseteq X$ is type-definable over M.

(ii) Suppose C is a compact space, and $f : X \to C$ a map. We say that f is *definable over* M, if for any closed subset D of C, $f^{-1}(D)$ is type-definable over M.

The following is well-known (see e.g. [6]).

Lemma 1.2. *Let X, E be as in Definition 1.1 and C be a compact space.*

(i) *The set X/E equipped with the logic topology is a compact space, and $\pi : X \to X/E$ is definable over M.*

(ii) *Conversely, suppose $f : X \to C$ is an M-definable map. Then:*