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On supercompactness and the continuum function [☆]

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ABSTRACT

Given a cardinal κ that is λ-supercompact for some regular cardinal $\lambda \geqslant \kappa$ and assuming GCH, we show that one can force the continuum function to agree with any function $F : [\kappa, \lambda] \cap \mathrm{REG} \to \mathrm{CARD}$ satisfying $\forall \alpha, \beta \in \mathrm{dom}(F)$ $\alpha < \mathrm{cf}(F(\alpha))$ and $\alpha < \beta \implies F(\alpha) \leqslant F(\beta)$, while preserving the λ-supercompactness of κ from a hypothesis that is of the weakest possible consistency strength, namely, from the hypothesis that there is an elementary embedding $j : V \to M$ with critical point κ such that $M^\lambda \subseteq M$ and $j(\kappa) > F(\lambda)$. Our argument extends Woodin's technique of surgically modifying a generic filter to a new case: Woodin's key lemma applies when modifications are done on the range of j, whereas our argument uses a new key lemma to handle modifications done off of the range of j on the ghost coordinates. This work answers a question of Friedman and Honzik [5]. We also discuss several related open questions.

1. Introduction

The behavior of the continuum function $\gamma \mapsto 2^\gamma$ on the regular cardinals was shown, by Easton, to be highly undetermined by the axioms of ZFC. Easton proved [3] that if F is any function from the regular cardinals to the cardinals satisfying $\alpha < \mathrm{cf}(F(\alpha))$ and $\alpha < \beta \implies F(\alpha) \leqslant F(\beta)$, then there is a cofinality-preserving forcing extension in which $2^\gamma = F(\gamma)$ for every regular cardinal γ. Large cardinal axioms impose additional restrictions on the continuum function on the regular cardinals. For example,

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if κ is a supercompact cardinal and GCH holds below κ, then GCH holds everywhere. It therefore seems natural to address the question: What functions can be forced to coincide with the continuum function on the regular cardinals while preserving large cardinals? From what hypotheses? In particular, let us consider the following question.

Question 1. Given a λ-supercompact cardinal κ where $\lambda \geqslant \kappa$ is a regular cardinal, and assuming GCH, what functions F from the regular cardinals to the cardinals can be forced to equal the continuum function on the interval $[\kappa, \lambda]$ while preserving the λ-supercompactness of κ and preserving cardinals? From what hypotheses?

Menas [11] proved that, assuming GCH, one can force the continuum function to agree at every regular cardinal with any *locally definable*[3] function F satisfying the requirements of Easton's theorem, while preserving all cofinalities and preserving the supercompactness of a cardinal κ. If κ is a Laver-indestructible supercompact cardinal in a model V, as in [9], it easily follows that one can force over this model to achieve any reasonable behavior of the continuum function at and above κ, while preserving the supercompactness of κ. In particular, starting with a Laver-indestructible supercompact cardinal, one can obtain a model with a measurable cardinal at which GCH fails. However, one can also obtain a model with a measurable cardinal at which GCH fails from a much weaker large cardinal assumption. Woodin proved that the existence of a measurable cardinal at which GCH fails is equiconsistent with the existence of an elementary embedding $j : V \to M$ with critical point κ such that $M^\kappa \subseteq M$ and $j(\kappa) > \kappa^{++}$ (see [2, Theorem 25.1]). Woodin's argument illustrates that under certain conditions, one may perform a type of surgical modification on a generic filter g to obtain g^* in order to meet the lifting criterion, $j''G \subseteq g^*$, and such that g^* remains generic. In Woodin's proof, the modifications made to g in order to obtain g^* only occur on the range of j, and his key lemma shows that such changes are relatively mild in the sense that for a given condition $p \in g$, the set over which modifications are made to obtain $p^* \in g^*$ has size at most κ. Hamkins showed [7] that Woodin's method could be applied to obtain an indestructibility theorem for tall cardinals.

The first author extended Woodin's surgery method to the case of partially supercompact cardinals in [1]. It is shown in [1] that the existence of a λ-supercompact cardinal κ such that $2^\lambda \geqslant \lambda^{++}$ is equiconsistent with the following hypothesis.

($*$) There is an elementary embedding $j : V \to M$ with critical point κ such that $M^\lambda \subseteq M$ and $j(\kappa) > \lambda^{++}$.

The method used in [1] is to, after a suitable preparatory iteration, blow up the size of the powerset of κ using Cohen forcing, in order to achieve $2^\kappa = \lambda^{++}$ and then use Woodin's method of surgery to lift the elementary embedding. Thus, one obtains a model in which κ is λ-supercompact and GCH fails at λ, *because* $2^\kappa = \lambda^{++}$. Answering a question posed in [1], Friedman and Honzik [5] used a variant of Sacks forcing for uncountable cardinals to show, from the hypothesis ($*$), one can obtain a forcing extension in which κ is λ-supercompact, GCH holds on $[\kappa, \lambda)$, and $2^\lambda \geqslant \lambda^{++}$. The methods of both [1] and [5] leave open the following question, which appears in [5]. Assuming GCH and ($*$), where $\kappa < \gamma < \lambda$ are regular cardinals, is there a cofinality-preserving forcing extension in which κ remains λ-supercompact, GCH holds on the interval $[\kappa, \gamma)$, and $2^\gamma = \lambda^{++}$? In this article we answer this question, and indeed, provide a full answer to Question 1, by proving the following theorem.

Theorem 1. *Suppose* GCH *holds,* $\kappa < \lambda$ *are regular cardinals, and* $F : [\kappa, \lambda] \cap \mathrm{REG} \to \mathrm{CARD}$ *is a function such that* $\forall \alpha, \beta \in \mathrm{dom}(F)$ $\alpha < \mathrm{cf}(F(\alpha))$ *and* $\alpha < \beta \Longrightarrow F(\alpha) \leqslant F(\beta)$. *If there is an elementary embedding*

[3] A function F is *locally definable* if there is a sentence ψ, true in V, and a formula $\varphi(x, y)$ such that for all cardinals γ, if $H_\gamma \models \psi$, then F has a closure point at γ and for all $\alpha, \beta < \gamma$, we have $F(\alpha) = \beta \leftrightarrow H_\gamma \models \varphi(\alpha, \beta)$.