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Theories without the tree property of the second kind

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ABSTRACT

We initiate a systematic study of the class of theories without the tree property of the second kind — NTP_2. Most importantly, we show: the burden is "sub-multiplicative" in arbitrary theories (in particular, if a theory has TP_2 then there is a formula with a single variable witnessing this); NTP_2 is equivalent to the generalized Kim's lemma and to the boundedness of ist-weight; the dp-rank of a type in an arbitrary theory is witnessed by mutually indiscernible sequences of realizations of the type, after adding some parameters — so the dp-rank of a 1-type in any theory is always witnessed by sequences of singletons; in NTP_2 theories, simple types are co-simple, characterized by the co-independence theorem, and forking between the realizations of a simple type and arbitrary elements satisfies full symmetry; a Henselian valued field of characteristic $(0, 0)$ is NTP_2 (strong, of finite burden) if and only if the residue field is NTP_2 (the residue field and the value group are strong, of finite burden respectively), so in particular any ultraproduct of p-adics is NTP_2; adding a generic predicate to a geometric NTP_2 theory preserves NTP_2.

Introduction

The aim of this paper is to initiate a systematic study of theories without the tree property of the second kind, or NTP_2 theories. This class was defined by Shelah implicitly in [31] in terms of a certain cardinal invariant κ_{inp} (see Section 2) and explicitly in [30], and it contains both simple and NIP theories. There was no active research on the subject until the recent interest in generalizing methods and results of stability theory to larger contexts, necessitated for example by the developments in the model theory of important algebraic examples such as algebraically closed valued fields [17].

We give a short overview of related results in the literature. The invariant κ_{inp}, the upper bound for the number of independent partitions, was considered by Tsuboi in [37] for the case of stable theories. In [2]

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Adler defines burden, by relativizing κ_{inp} to a fixed partial type, makes the connection to weight in simple theories and defines strong theories. Burden in the context of NIP theories, where it is called dp-rank, was already introduced by Shelah in [33] and developed further in [28,23,22]. Results about forking and dividing in NTP_2 theories were established in [8]. In particular, it was proved that a formula forks over a model if and only if it divides over it (see Section 4). Some facts about ordered inp-minimal theories and groups (that is with $\kappa^1_{\mathrm{inp}} = 1$) are proved in [14,36]. In [19, Theorem 4.13] Ben Yaacov shows that if a structure has IP, then its randomization (in the sense of continuous logic) has TP_2. Malliaris [27] considers TP_2 in relation to the saturation of ultra-powers and the Keisler order. In [5] Chatzidakis observes that ω-free PAC fields have TP_2.

A brief description of the results in this paper.

In Section 2 we introduce inp-patterns, burden, establish some of their basic properties and demonstrate that burden is sub-multiplicative: that is, if $\mathrm{bdn}(a/C) < \kappa$ and $\mathrm{bdn}(b/aC) < \lambda$, then $\mathrm{bdn}(ab/C) < \kappa \times \lambda$. As an application we show that the value of the invariant of a theory $\kappa_{\mathrm{inp}}(T)$ does not depend on the number of variables used in the computation. This answers a question of Shelah from [31] and shows in particular that if T has TP_2, then some formula $\phi(x, y)$ with x a singleton has TP_2. It remains open whether burden in NTP_2 theories is actually sub-additive.

In Section 3 we describe the place of NTP_2 in the classification hierarchy of first-order theories and the relationship of burden to dp-rank in NIP theories and to weight in simple theories. We also recall some combinatorial "structure/non-structure" dichotomy due to Shelah, and discuss the behavior of the SOP_n hierarchy restricting to NTP_2 theories.

Section 4 is devoted to forking (and dividing) in NTP_2 theories. After discussing strictly invariant types, we give a characterization of NTP_2 in terms of the appropriate variants of Kim's lemma, local character and bounded weight relatively to strict non-forking. As an application we consider theories with dependent dividing (i.e. whenever $p \in S(N)$ divides over $M \prec N$, there some $\phi(x, a) \in p$-dividing over M and such that $\phi(x, y)$ is NIP) and show that any theory with dependent dividing is NTP_2. Finally we observe that the analysis from [8] generalizes to a situation when one is working inside an NTP_2 type in an arbitrary theory.

A famous equation of Shelah "NIP = stability+dense linear order" turned out to be a powerful ideological principle, at least at the early stages of the development of NIP theories. In this paper the equation "NTP_2 = simplicity + NIP" plays an important role. In particular, it seems very natural to consider two extremal kinds of types in NTP_2 theories (and in general) — simple types and NIP types. While it is perfectly possible for an NTP_2 theory to have neither, they form important special cases and are not entirely understood.

In Section 5 we look at NIP types. In particular we show that the results of the previous section on forking localized to a type combined with honest definitions from [9] allow to omit the global NTP_2 assumption in the theorem of [22], thus proving that dp-rank of a type in arbitrary theory is always witnessed by mutually indiscernible sequences of its realizations, after adding some parameters (see Theorem 5.3). We also observe that in an NTP_2 theory, a type is NIP if and only if every extension of it has only boundedly many global non-forking extensions.

In Section 6 we consider simple types (defined as those types for which every completion satisfies the local character), first in arbitrary theories and then in NTP_2. While it is more or less immediate that on the set of realizations of a simple type forking satisfies all the properties of forking in simple theories, the interaction between the realizations of a simple type and arbitrary tuples seems more intricate. We establish full symmetry between realizations of a simple type and arbitrary elements, answering a question of Casanovas in the case of NTP_2 theories (showing that simple types are co-simple, see Definition 6.7). Then we show that simple types are characterized as those satisfying the co-independence theorem and that co-simple stably embedded types are simple (so in particular a theory is simple if and only if it is NTP_2 and satisfies the independence theorem).