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The principle of signature exchangeability

Tahel Ronel [1], Alena Vencovská [*,2]

School of Mathematics, The University of Manchester, Manchester M13 9PL, United Kingdom

A R T I C L E I N F O

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A B S T R A C T

We investigate the notion of a *signature* in Polyadic Inductive Logic and study the probability functions satisfying the *Principle of Signature Exchangeability*. We prove a representation theorem for such functions on binary languages and show that they satisfy a binary version of the Principle of Instantial Relevance. We discuss polyadic versions of the Principle of Instantial Relevance and Johnson's Sufficientness Postulate.

1. Introduction

This paper is set in Pure Inductive Logic (PIL), see for example [10], which the reader is referred to for background[3] and an extensive bibliography. In this subject, we are concerned with assigning subjective probabilities to sentences of a language according to rational considerations, traditionally based on the notions of symmetry, relevance and irrelevance.

The principle of Constant Exchangeability (Ex) or in Carnap's terms, the Axiom of Symmetry [1,3], is a widely accepted and commonly assumed rational requirement in Pure Inductive Logic. Informally, this is the statement that in the absence of further knowledge, different individuals of our universe should be treated equally. In the usual framework of Inductive Logic it means that the probability assigned to a sentence is independent of the particular constants instantiating it. In addition, in the thoroughly studied unary context, this principle exists in an equivalent formulation – as invariance under signatures of state

* Corresponding author.
E-mail addresses: tahel.ronel@manchester.ac.uk (T. Ronel), alena.vencovska@manchester.ac.uk (A. Vencovská).

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3 In particular, the goals and methods of PIL are clearly illustrated in the first short chapter of [10].

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descriptions. This unary characterisation of the principle has led to some of the most significant results in Unary Inductive Logic thus far. These include, for example, a complete characterisation of functions satisfying Ex, and the Principle of Instantial Relevance (see page 19) following as a logical consequence of Constant Exchangeability.

In contrast, such results have so far not translated satisfactorily into the polyadic. Having extended the concept of atoms to polyadic languages (see [10,12]), in this account we generalise the notion of a signature to polyadic Inductive Logic and investigate the theory this yields for higher arity languages. We begin by giving a brief account of the unary portion we shall be concerned with for the purpose of this paper, then suggest new methods and formulations for these concepts for general polyadic languages. Specifically, we present a polyadic definition of a signature and a principle of invariance under this notion, an independence principle characterising the basic functions satisfying this new signature-based principle, and polyadic versions of the Principle of Instantial Relevance and Johnson's Sufficientness Postulate. We present this initially for languages with at most binary relation symbols and then, in Section 3 of the paper, we focus on the general case.

The context of this paper is as follows. We work with a first order language L containing finitely many relation symbols $R_1, \dots, R_q$ of arities $r_1, \dots, r_q$ respectively and countably many constant symbols $a_1, a_2, a_3, \dots$ (which are intended to exhaust the universe), using the usual logical connectives and quantifiers. SL denotes the set of all sentences of the language L and $QFSL$ the set of all quantifier-free sentences of the language. $b_1, \dots, b_n$ or sometimes also $b'_1, \dots, b'_n$ are used to denote some distinct constants from amongst the $a_1, a_2, \dots$, and S_n stands for the set of permutations of $\{1, 2, \dots, n\}$.

We say that a language is *unary* if it contains only unary predicate symbols; it is *r-ary* if all its relation symbols are at most r-ary and at least one is r-ary. If $r = 2$, we say *binary* rather than 2-ary.

Definition. A function $w : SL \to [0, 1]$ is a *probability function* if for all θ, ϕ and $\exists x\, \psi(x) \in SL$

(P1) If θ is logically valid then $w(\theta) = 1$.
(P2) If θ and ϕ are mutually exclusive then $w(\theta \vee \phi) = w(\theta) + w(\phi)$.
(P3) $w(\exists x\, \psi(x)) = \lim_{n\to\infty} w(\psi(a_1) \vee \psi(a_2) \vee \dots \vee \psi(a_n))$.

Probability functions have a number of desirable properties, see for example [10, Chapter 3]; note in particular that logically equivalent sentences always get the same probability. We will be interested in sentences and formulae only up to logical equivalence, and somewhat abusing notation we will often use '$=$' in place of '$\equiv$'.

The conditional probability of θ given ϕ, for ϕ such that $w(\phi) \neq 0$, is defined as follows:

$$w(\theta \mid \phi) \;=\; \frac{w(\theta \wedge \phi)}{w(\phi)}.$$

We adopt the convention that expressions like $w(\theta \mid \phi) = a$ stand for $w(\theta \wedge \phi) = a\, w(\phi)$ even if $w(\phi) = 0$.

Any w satisfying just (P1) and (P2) on the quantifier free sentences of L has a unique extension to a probability function on SL, see [6], so in many situations it suffices to think of probability functions as defined on quantifier-free sentences only, and satisfying (P1) and (P2).

As explained in [10, Chapter 7], this can be further reduced to a special class of such sentences called *state descriptions*, that is, to sentences $\Theta(b_1, \dots, b_m)$ of the form

$$\bigwedge_{i=1}^{q} \;\; \bigwedge_{\langle j_1 \dots, j_{r_i}\rangle \in \{1,\dots,m\}^{r_i}} \pm R_i(b_{j_1} \dots, b_{j_{r_i}}) \tag{1}$$