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Bunched sequential information

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ABSTRACT

It is known that the logic BI of bunched implications is a logic of resources. Many studies have reported on the applications of BI to computer science. In this paper, an extension BIS of BI by adding a sequence modal operator is introduced and studied in order to formalize more fine-grained resource-sensitive reasoning. By the sequence modal operator of BIS, we can appropriately express "sequential information" in resource-sensitive reasoning. A Gentzen-type sequent calculus SBIS for BIS is introduced, and the cut-elimination and decidability theorems for SBIS are proved. An extension of the Grothendieck topological semantics for BI is introduced for BIS, and the completeness theorem with respect to this semantics is proved. The cut-elimination, decidability and completeness theorems for SBIS and BIS are proved using some theorems for embedding BIS into BI.

1. Introduction

The *logic BI of bunched implications*, which was originally introduced by O'Hearn and Pym [23], is known to be a logic of resources. BI is defined as a combination of *multiplicative intuitionistic linear logic* [7] and *intuitionistic logic*. BI has a number of applications to computer science, such as resource distribution, Petri net specifications, memory allocation models, typing systems, programming languages and assertion languages for mutable data structures [8,22,25,10]. BI has also some mathematical foundations such as Kripke- and categorical-semantics, sequent calculi, natural deduction systems and tableaux calculi [5,23,24].

In this paper, an extension BIS of BI by adding a sequence modal operator is introduced and studied in order to formalize more fine-grained resource-sensitive reasoning. By the sequence modal operator of BIS, we can appropriately express "sequential information" in resource-sensitive reasoning. Some linear and temporal logics with the sequence modal operator have been introduced and studied in [16,17,13,14]. But, there was no extension of BI by adding the sequence modal operator. A Gentzen-type sequent calculus

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SBIS for BIS is introduced extending a Gentzen-type sequent calculus SBI for BI, and the cut-elimination and decidability theorems for SBIS are proved using a theorem for syntactically embedding SBIS into SBI. A semantics for SBIS is introduced extending the *Grothendieck topological semantics* for BI [6,24,25], and the completeness theorem with respect to this semantics is proved using two theorems for semantically and syntactically embedding BIS into BI.

The motivation underlying the use of the notion of "sequences" in the sequence modal operator is as follows. The notion of "sequences" is fundamental to practical reasoning in computer science because it can appropriately represent "data sequences," "time sequences," "action sequences" etc. The notion of sequences is thus useful to represent "ordered information," "additional information," "hierarchical information," "preference information," and "ontological information". Such information is called here "sequential information" which can be represented by sequences. This is suitable because a sequence structure gives a *monoid* $\langle M, ;, \emptyset \rangle$ with *informational interpretation* [31]:

1. M is a set of pieces of (ordered or prioritized) information (i.e., a set of sequences),
2. ; is a binary operator (on M) that combines two pieces of information (i.e., a concatenation operator on sequences),
3. $\emptyset$ is the empty piece of information (i.e., the empty sequence).

The sequence modal operator $[b]$ represents "sequential information" as labels. A formula of the form $[b_1 ; b_2 ; \cdots ; b_n]\alpha$, which is equivalent to $[b_1][b_2]\cdots[b_n]\alpha$, intuitively means that "α is true based on a sequence $b_1 ; b_2 ; \cdots ; b_n$ of (ordered or prioritized) information pieces." Further, a formula of the form $[\emptyset]\alpha$, which coincides with α, intuitively means that "α is true without any additional information."

The structure of this paper is summarized as follows.

In Section 2, a Gentzen-type sequent calculus for BIS is discussed. Firstly, a Gentzen-type sequent calculus SBI for BI is introduced. Secondly, a Gentzen-type sequent calculus SBIS for BIS is introduced extending SBI, and a theorem for syntactically embedding SBIS into SBI is proved. The cut-elimination and decidability theorems for SBIS are derived from this embedding theorem.

In Section 3, a semantics for BIS is discussed. Firstly, a Grothendieck topological semantics for BI is presented, and the completeness theorem with respect to this semantics is presented based on the original results by Pym et al. [6,24,25]. Secondly, an extension of the Grothendieck topological semantics is introduced for SBIS, and a theorem for semantically embedding SBIS into SBI is proved. The completeness theorem with respect to this semantics is proved combining the semantical embedding theorem and the syntactical embedding theorem.

In Section 4, some illustrative examples for BIS-formulas are presented. It is explained that sequential information (ordered, additional and hierarchical information) can suitably be expressed using BIS-formulas.

In Section 5, some conclusions and remarks are addressed.

2. Proof systems

2.1. Preliminaries

The language used is introduced below. *Formulas* of BI are constructed from propositional variables, **1** (multiplicative constant), $\top, \bot$ (additive constants), $-\!\!*$ (linear or multiplicative implication), $\rightarrow$ (intuitionistic or additive implication), $\wedge$ (additive conjunction), $*$ (multiplicative conjunction), and $\vee$ (additive disjunction). Lower-case letters $p, q, \ldots$ are used to represent propositional variables, and Greek lower-case letters $\alpha, \beta, \ldots$ are used to represent formulas. We write $A \equiv B$ to indicate the syntactical identity between A and B. Since all logics discussed in this paper are formulated as sequent calculi, we will sometimes identify a sequent calculus with the logic determined by it.