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Chaos and indecomposability

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ABSTRACT

We use recent developments in local entropy theory to prove that chaos in dynamical systems implies the existence of complicated structure in the underlying space. Earlier Mouron proved that if X is an arc-like continuum which admits a homeomorphism f with positive topological entropy, then X contains an indecomposable subcontinuum. Barge and Diamond proved that if G is a finite graph and $f : G \to G$ is any map with positive topological entropy, then the inverse limit space $\varprojlim(G, f)$ contains an indecomposable continuum. In this paper we show that if X is a G-like continuum for some finite graph G and $f : X \to X$ is any map with positive topological entropy, then $\varprojlim(X, f)$ contains an indecomposable continuum. As a corollary, we obtain that in the case that f is a homeomorphism, X contains an indecomposable continuum. Moreover, if f has uniformly positive upper entropy, then X is an indecomposable continuum. Our results answer some questions raised by Mouron and generalize the above mentioned work of Mouron and also that of Barge and Diamond. We also introduce a new concept called zigzag pair which attempts to capture the complexity of a dynamical systems from the continuum theoretic perspective and facilitates the proof of the main result.

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1. Introduction

During the last thirty years or so, many interesting connections between topological dynamics and continuum theory have been established. One of the underlying themes is that somehow complicated dynamics should imply existence of complicated continua. More precisely, if X is a continuum (i.e., compact connected metric space) and $h : X \to X$ is a homeomorphism of X such that (X, h) is chaotic in some sense, then X should contain a complicated subcontinuum. In the case that h happens to be simply a continuous surjection, then the inverse limit space $\varprojlim(X, h)$ should contain a complicated continuum.

This line of investigation began with the seminal paper of Barge and Martin [2] where they showed that if f is a continuous self-map of the interval $I = [0, 1]$ which has a periodic point not a power of 2, then $\varprojlim(I, f)$ contains an indecomposable continuum. By Misiurewicz's theorem [4], we have that a continuous map of the interval has a periodic point not a power of 2 if and only if the map has positive topological entropy. Hence, we have that if $f : I \to I$ has positive topological entropy, then $\varprojlim(I, f)$ contains an indecomposable continuum. As one of the generally accepted definitions of chaos is that the map has positive topology entropy, we have, in the case of the interval, that chaos implies the existence of a complicated subcontinuum in the inverse limit space.

Let G be a compact metric space. A continuous mapping g from X onto G is an *ε-mapping* if for every $x \in X$, the diameter of $g^{-1}(x)$ is less than ε. A continuum X is *G-like* if for every $\varepsilon > 0$ there is a ε-mapping from X onto G. Or, equivalently, X is homeomorphic to the inverse limit of a sequence of continuous surjections of G. (See Section 2 for definitions.) In particular, arc-like continua are those which are G-like for $G = [0, 1]$.

Ingram [10] and Ye [25], independently, showed that if X is an arc-like continuum and $f : X \to X$ is a continuous map with a periodic point not a power of 2, then $\varprojlim(X, f)$ contains an indecomposable continuum. As of now it is not known if positive entropy maps of arc-like hereditarily decomposable continua must have a periodic point which is not a power of 2. Hence, one cannot conclude directly from their result that positive entropy on arc-like continua implies indecomposability in the corresponding inverse limit space. Ye in the same paper and also in [17] proved that for certain special types of homeomorphisms of the arc-like continua, positive entropy implies that the domain contains an indecomposable continuum. In 2011, Mouron [21] settled the more general problem by proving that if h is a homeomorphism of an arc-like continuum and h has positive entropy, then X contains an indecomposable continua. He raised the questions if the same result holds for the monotone open maps of arc-like continua or, in general, for monotone maps of arc-like continua.

Dynamical systems where X is more general than arc-like continua also have been investigated from this perspective. In 1990 Seidler [24] showed that if X is a continuum which admits a homeomorphism of positive topological entropy, then X is not a regular continuum. A continuum is *regular* if for every $\varepsilon > 0$, there is an $n \in \mathbb{N}$ such that X