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Alternating knots with unknotting number one

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We show that an alternating knot with unknotting number one has an unknotting crossing in any alternating diagram. We also prove that an alternating knot has unknotting number one if and only if its branched double cover arises as half-integer surgery on a knot in S^3, thus establishing a converse to the Montesinos trick. Along the way, we reprove a characterisation of almost-alternating diagrams of the unknot originally due to Tsukamoto. These results are established using the obstruction to unknotting number one developed by Greene.

1. Introduction

Given a knot $K \subset S^3$, its *unknotting number*, $u(K)$, is a classical knot invariant going back to the work of Tait in the 19th century [39]. It is defined to be the minimal number of crossing changes required in any diagram of K to obtain the unknot. Upper bounds for the unknotting number are easy to obtain, since one can take some diagram and find a sequence of crossing changes giving the unknot. It is far harder to establish effective lower bounds for the unknotting number, as it is not generally known which diagrams will exhibit the actual unknotting number [1,2,18]. One classical lower bound

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is the signature of a knot as defined by Trotter [42], which satisfies $|\sigma(K)| \leq 2u(K)$ [28]. This is a particularly useful bound, since it may be computed in a variety of ways [13,42]. Other bounds and obstructions have been constructed through the use of various knot-theoretic and topological invariants, including, among others, the Alexander module [23, Theorem 7.10], the Jones polynomial [38], and the intersection form of 4-manifolds [8,31].

The case of unknotting number one has been particularly well-studied. Recall that a *minimal diagram* for a knot is one containing the minimal possible number of crossings. Kohn made the following conjecture regarding unknotting number one knots and their minimal diagrams.

Kohn's Conjecture *(Conjecture 12, [21]).* If K is a knot with $u(K) = 1$, then it has an unknotting crossing in some minimal diagram.

This has been resolved in a number of cases. The two-bridge knots with unknotting number one were classified by Kanenobu and Murakami [19], using the Cyclic Surgery Theorem [10]. For alternating large algebraic knots, the conjecture was settled by Gordon and Luecke [14]. Most recently, the conjecture was proved for alternating 3-braid knots by Greene [16], using a refined version of obstructions first developed from Heegaard Floer homology by Ozsváth and Szabó [33].

Our main result is the following.

Theorem 1. *For an alternating knot, K, the following are equivalent:*

(i) $u(K) = 1$*;*
(ii) *The branched double cover, $\Sigma(K)$, can be obtained by half-integer surgery on a knot in S^3;*
(iii) *K has an unknotting crossing in every alternating diagram.*

Since the minimal diagrams of alternating knots are alternating, this resolves Kohn's conjecture for alternating knots [20,29,40].

In general, Kohn's Conjecture seems somewhat optimistic. For example, there are 14-crossing knots with unknotting number one and minimal diagrams not containing an unknotting crossing [37]. However these examples are not sufficient to disprove the conjecture, since they still all possess some minimal diagram with an unknotting crossing.

Theorem 1 can be interpreted as showing that understanding alternating knots with unknotting number one is equivalent to understanding almost-alternating diagrams of the unknot, where an *almost-alternating diagram* is one which is obtained from an alternating diagram by a single crossing change. A result of Tsukamoto shows that any reduced almost-alternating diagram of the unknot can be built up using only certain types of isotopies: *flypes*, which are illustrated in Fig. 2; and *tongue* and *twirl* moves, which are