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On conjugacy of Cartan subalgebras in extended affine Lie algebras

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ABSTRACT

That finite-dimensional simple Lie algebras over the complex numbers can be classified by means of purely combinatorial and geometric objects such as Coxeter–Dynkin diagrams and indecomposable irreducible root systems, is arguably one of the most elegant results in mathematics. The definition of the root system is done by fixing a Cartan subalgebra of the given Lie algebra. The remarkable fact is that (up to isomorphism) this construction is independent of the choice of the Cartan subalgebra. The modern way of establishing this fact is by showing that all Cartan subalgebras are conjugate.
For symmetrizable Kac–Moody Lie algebras, with the appropriate definition of Cartan subalgebra, conjugacy has been established by Peterson and Kac. An immediate consequence of this result is that the root systems and generalized Cartan matrices are invariants of the Kac–Moody Lie algebras. The purpose of this paper is to establish conjugacy of Cartan subalgebras for extended affine Lie algebras; a natural class

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of Lie algebras that generalizes the finite-dimensional simple Lie algebra and affine Kac–Moody Lie algebras.

Introduction

Let $\mathfrak{g}$ be a finite-dimensional split simple Lie algebra over a field k of characteristic 0, and let $\mathbf{G}$ be the simply connected Chevalley–Demazure algebraic group associated to $\mathfrak{g}$. Chevalley's theorem [11, VIII, §3.3, Cor de la Prop. 10] asserts that all split Cartan subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ are conjugate under the adjoint action of $\mathbf{G}(k)$ on $\mathfrak{g}$. This is one of the central results of classical Lie theory. One of its immediate consequences is that the corresponding root system is an invariant of the Lie algebra (i.e., it does not depend on the choice of Cartan subalgebra).

We now look at the analogous question in the infinite dimensional setup as it relates to extended affine Lie algebras (EALAs for short). We assume henceforth that k is algebraically closed, but the reader should keep in mind that our results are more akin to the setting of Chevalley's theorem for general k than to conjugacy of Cartan subalgebras in finite-dimensional simple Lie algebras over algebraically closed fields. The role of $(\mathfrak{g}, \mathfrak{h})$ is now played by a pair (E, H) consisting of a Lie algebra E and a "Cartan subalgebra" H. There are other Cartan subalgebras, and the question is whether they are conjugate and, if so, under the action of which group.

The first example is that of untwisted affine Kac–Moody Lie algebras. Let $R = k[t^{\pm 1}]$. Then

$$E = \mathfrak{g} \otimes_k R \oplus kc \oplus kd \tag{0.0.1}$$

and

$$H = \mathfrak{h} \otimes 1 \oplus kc \oplus kd. \tag{0.0.2}$$

The relevant information is as follows. The k-Lie algebra $\mathfrak{g} \otimes_k R \oplus kc$ is a central extension (in fact the universal central extension) of the k-Lie algebra $\mathfrak{g} \otimes_k R$. The derivation d of $\mathfrak{g} \otimes_k R$ corresponds to the degree derivation td/dt acting on R. Finally $\mathfrak{h}$ is a fixed Cartan subalgebra of $\mathfrak{g}$. The nature of H is that it is abelian, it acts k-diagonalizably on E, and it is maximal with respect to these properties. Correspondingly, these algebras are called MADs (Maximal Abelian Diagonalizable) subalgebras. A celebrated theorem of Peterson and Kac [24] states that all MADs of E are conjugate (under the action of a group that they construct which is the analogue of the simply connected group in the