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Quandle cocycles from invariant theory

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Abstract

Let G be a group. Any G-module M has an algebraic structure called a G-family of Alexander quandles. Given a 2-cocycle of a cohomology associated with this G-family, topological invariants of (handlebody) knots in the 3-sphere are defined. We develop a simple algorithm to algebraically construct n-cocycles of this G-family from G-invariant group n-cocycles of the abelian group M. We present many examples of 2-cocycles of these G-families using facts from (modular) invariant theory.

Keywords: Group homology; Quandle; Invariant theory; Knots

1. Introduction

A quandle is a set with a binary operation whose definition was motivated by knot theory. A particularly interesting class is the quandle associated with the G-family of Alexander quandles proposed by Ishii et al. [9]. Specifically, each quandle in the class is defined as the product $M \times G$ of a group G and a right G-module M with the binary operation

$$(M \times G) \times (M \times G) \longrightarrow M \times G, \qquad (a, g, b, h) \longmapsto (\,(a-b)\cdot h + b,\ h^{-1}gh\,). \quad (1)$$

Given a quandle X, Fenn et al. defined the rack space BX [6] by analogy to the classifying spaces of groups. By slightly modifying its cohomology $H^*(BX)$, Carter et al. introduced quandle cohomologies $H^*_Q(X; A)$ [4,5]. Furthermore, if X is of finite order, they used 2- and 3-cocycles to define quandle cocycle invariants of links in the 3-sphere ($L \subset S^3$) and of knotted

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surfaces in the 4-sphere ($\Sigma_g \hookrightarrow S^4$). Ishii et al. showed that if X is a quandle of the form $M \times G$ above and its quandle 2-cocycle satisfies certain strong conditions, then the cocycle invariant is generalized for handlebody knots in the 3-sphere [9].

However, there are few methods for finding cocycles in the quandle cohomology $H_Q^*(X; A)$ compared with group cohomology theory. Here we refer to three results. First, for any quandle X, Inoue and Kabaya constructed a map from the homology $H_*(BX)$ to a simplicial homology of X to interpret the Chern–Simons class as a quandle cocycle [7]. Second, most known quandle 3-cocycles were obtained by Mochizuki [12]. He determined all the 2- and 3-cocycles of some Alexander quandles, which are subquandles of the form $M \times \{1\} \subset M \times G$ with $G = \mathbb{Z}$ and $M = \mathbb{F}_q$ (note that all quandle cocycles in the original paper [4] are included in Mochizuki cocycles). Mochizuki's construction involved carefully solving all the cocycle conditions via differential equations. Third, using his method, I previously described all n-cocycles of all the Alexander quandles of prime order [14].

This paper deals with arbitrary groups G and right G-modules M and develops a simple algorithm to algebraically construct n-cocycles of the above quandle on $M \times G$ (Theorem 3.2). After a review of quandles in Section 2, we construct a chain map φ_n from the quandle complex of the quandle $M \times G$ to the G-coinvariants of the group complex of the abelian group M (Proposition 3.1). We define the map φ_n by modifying the Inoue–Kabaya map (Remark 3.3). Section 4 shows that pullback using the chain map φ_n permits the strong conditions in [9] mentioned above (Propositions 4.3 and 4.4). Section 4.2 gives simple formulae for such quandle 2-cocycles, even though it is not easy to obtain n-cocycles of the G-family even if G is abelian (cf. [8, Proposition 12] in the abelian case). In conclusion, if we find a G-invariant group n-cocycle of M, then we obtain a quandle n-cocycle as the pullback via the chain map φ_n (Theorem 3.2). This allows us to compute the associated cocycle invariants of tame links, of knotted surfaces, and of handlebody knots.

In Section 5, for the case in which G and M are of finite order, we seek G-invariant group n-cocycles of M based on known results from (modular) invariant theory. For example, for finite groups of the Lie type, Chern–Weil theory and the Dickson theorem produce many G-invariant multilinear maps (Examples 5.3, 5.8 and 5.10); since multilinear maps are group cocycles, we obtain many quandle cocycles via the chain map φ_n. We also address two standard representations of $G = SL(2; \mathbb{F}_p)$ and $G = O(3; \mathbb{F}_p)$; first we easily obtain a quandle 2-cocycle (Proposition 5.6). Using the Bockstein map, we discover quandle 3-cocycles (Propositions 5.11 and 5.12) and show that they are not null-cohomologous (Propositions 5.6 and 5.13 and Remark 5.14). We also observe a relation between the chain map φ_* and the Mochizuki 3-cocycles mentioned above (Example 5.1).

Thus, using invariant theory, we can explicitly obtain more examples of n-cocycles of the quandles $M \times G$. Our results should provide the motivation to find G-invariant polynomials. The quandle cocycles will be applicable in the study of knot theory. In fact, the cocycle invariant constructed by Ishii et al. from the cocycle in Proposition 5.4 shows that several handlebody knots are not equivalent to their mirror images [9, Table 2].

2. Review of quandles and quandle homologies

In this section, we review G-families of quandles and quandle homologies.

A *quandle* X is a non-empty set with a binary operation $(x, y) \to x \lhd y$ such that, for any $x, y, z \in X$, $x \lhd x = x$, $(x \lhd y) \lhd z = (x \lhd z) \lhd (y \lhd z)$ and there exists uniquely $w \in X$ satisfying $w \lhd y = x$ [10,11]. For example, a $\mathbb{Z}[T^{\pm}]$-module M has a quandle structure given