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# Weakly compressible flow through a cylinder with pressure-dependent viscosity and Navier-slip at the wall

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## HIGHLIGHTS

- Compressible Navier–Stokes flow with pressure-dependent viscosity and Navier-slip.
- Second-order perturbation solution with compressibility as perturbation parameter.
- Viscosity–density parameter at second order for axial velocity, density, flow rate.
- Viscosity–density parameter at first order for pressure, Darcy friction factor.
- Darcy friction factor, pressure drop may increase due to viscosity–pressure dependence.

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ABSTRACT

A perturbation solution is derived for weakly compressible force-driven flow through a cylinder of fixed cross-section, with pressure-dependent viscosity and Navier-slip at the wall. The compressibility number is used as the perturbation parameter and a solution is derived up to second order. The perturbation solution is used to assess changes in flow for different Reynolds numbers, Froude numbers, aspect ratios, compressibilities, pressure-dependent viscosities and slip values. The viscosity–pressure dependence is controlled by compressibility and by a viscosity–density parameter. The following results are found: the viscosity–density parameter appears only at second-order for axial velocity, density and average volumetric flow rate, while at first order for pressure, average pressure drop and average Darcy friction factor; the volumetric flow rate decreases with increasing compressibility if forcing is small enough, increases with slip, and decreases with viscosity–density parameter; the average Darcy friction factor and average pressure drop can increase with increasing compressibility if the viscosity–density parameter is large enough. Although these results are as expected, the key contribution of this paper is the derivation of analytical expressions for the flow that incorporate the pressure dependence effects.



## 1. Introduction

Many engineering models consider flow of an incompressible fluid subject to the no-slip boundary condition at the walls of the flow domain. In recent years, the effect of compressibility and wall slip on flow has gained popularity.

Applications for compressible flows that can still be treated as isothermal Newtonian fluids include high pressure flows encountered in polymer and food processing, pumping of crude oil and fuel oil, fluid film lubrication, microfluidics, pharmaceutical tablet manufacturing and some geophysical flows [1–4]. Flows can involve high speeds such as in high-speed water jet cutting [5], as well as low-speed flows involving long pipelines [6], extrusion processes [7] and extrudate swell flows [8–10]. Wall slip has historically been used primarily for complex fluids such as polymer melts or concentrated solutions [11], but current, more sophisticated measuring devices have created a refound interest in the subject, especially in the fields of microfluidic and micro-electromechanical devices [12]. Experimental verification of slip for Newtonian fluids is reviewed in [13].

In some applications, although the viscosity can become density-dependent, the flow can still be treated as incompressible since the effect of pressure is much larger on the viscosity than on the mass density [11,14]. However, in waxy crude oil transport, polymer extrusion and polymer injection molding for example, the flow has to be treated as compressible [6,15–17]. Additionally, particle-based methods used for flow simulations are compressible in nature, and theoretical calculations for the viscosity give rise to density-dependent viscosity expressions [18] that

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can lead to changes in the velocity profile even in simple flow geometries [19–25]. The combination of slip and compressibility is very important in microfluidic devices in microelectromechanical systems (MEMS) [26,27].

Compressible flow models require an equation of state that relates pressure to density. The ideal gas equation of state is frequently used [20,23,21], or a more general linear form thereof [22,24,25]. Density-dependent viscosity expressions can be found in studies for groundwater flow [28], for polytropic gases [29], in shallow water theory [30], in self-gravitating fluids [31] and in particle-based methods such as Multiparticle Collision Dynamics (MPC) [32,18]. Exact analytical solutions are difficult to attain in these cases due to the high nonlinearity of the flow equations, however some perturbation solutions have been presented in the literature. In [24], a perturbation solution for a weakly compressible no-slip flow with compressibility number as perturbation parameter was solved to second order. A similar perturbation method was used for an approximate analytical solution for flow through an annulus [33], non-Newtonian flow [34,35] and Newtonian flow with Navier-slip at the wall [22]. The only cases where an analytical solution was found was for incompressible flow with pressure-dependent viscosity and no-slip at the wall [36–38], as well as for a viscoelastic fluid [39]. Finally, an asymptotic solution for compressible no-slip flow with pressure-dependent viscosity was considered in [40]. No publications have been made for a pressure-dependent viscosity with Navier-slip at the wall that provide analytical expressions incorporating pressure–density effects.

In this paper, a perturbation solution is presented for the case of compressible flow with Navier-slip at the wall subject to a pressure-dependent viscosity and a linear equation of state. The compressibility parameter arising from non-dimensionalizing the equation of state is used as the perturbation parameter. Although [40] considers a pressure-dependent viscosity, their perturbation solution has two perturbation parameters: one from the equation of state, the other from the pressure-dependent viscosity. In our work, we have only one perturbation parameter arising from the equation of state. Additionally, [40] uses the no-slip boundary condition whereas we use the Navier-slip condition instead, and we have a forcing term not used in [40]. Finally, we did not impose conditions on the derivatives for the radial velocity as imposed in [40] to determine the second-order radial velocity solution. The motivation for adding a forcing term to the system is due to the fact that some particle-based methods, in particular multiparticle collision dynamics, can exhibit artifacts if the flow is pressure-driven [41] whereas good agreement between the theoretically-predicted velocity profile can be achieved when a forcing term is used instead [42,43]. Furthermore, the theoretically predicted viscosity for such models is density-dependent ([32,18], Appendix C).

This paper is organized as follows: In Section 2 we present the governing equations, the boundary conditions, and the non-dimensionalized problem. The second-order perturbation solution is derived in Section 3 with corresponding results presented in Section 4. Important conclusions and future work are contained in the last Section 5.

## 2. Governing equations

The governing equations of motion for a steady compressible Newtonian fluid with variable viscosity are given by [44,45]

$$\rho\,(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P + \nabla\cdot\tau + \mathbf{F} \tag{1}$$

$$\tau = \mu\left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right) - \frac{2}{3}\mu(\nabla\cdot\mathbf{v})\mathbf{I} \tag{2}$$

$$\nabla\cdot(\rho\mathbf{v}) = 0 \tag{3}$$

where $\rho$ is the density, $\mathbf{v}$ is the velocity vector, $p$ is the pressure, $\mathbf{F}$ is an external force, $\tau$ is the shear stress tensor, $\mu$ is the viscosity, $\nabla$ is the gradient operator for the spatial coordinates $x$, $y$ and $z$, and $\mathbf{I}$ is the identity tensor. In this work, it is assumed that the coefficient of bulk viscosity $\kappa$ is negligible.

In cylindrical coordinates for flow with axial symmetry, and $\mathbf{v} = (u_r, u_\theta, u_z) = (v(r,z), 0, u(r,z))$, the $\theta$-momentum equation is satisfied identically and the remaining momentum and mass conservation equations become

$$\begin{aligned}\rho\,(\mathbf{v}\cdot\nabla)\,v \;=\; & -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 v - \frac{v}{r^2}\right) + \frac{\mu}{3}\frac{\partial}{\partial r}(\nabla\cdot\mathbf{v})\\ & + 2\frac{\partial\mu}{\partial r}\frac{\partial v}{\partial r} - \frac{2}{3}(\nabla\cdot\mathbf{v})\frac{\partial\mu}{\partial r} + \frac{\partial\mu}{\partial z}\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right)\end{aligned} \tag{4}$$

$$\begin{aligned}\rho\,(\mathbf{v}\cdot\nabla)\,u \;=\; & -\frac{\partial p}{\partial z} + \mu\nabla^2 u + \frac{\mu}{3}\frac{\partial}{\partial z}(\nabla\cdot\mathbf{v})\\ & + 2\frac{\partial\mu}{\partial z}\frac{\partial u}{\partial z} - \frac{2}{3}(\nabla\cdot\mathbf{v})\frac{\partial\mu}{\partial z}\\ & + \frac{\partial\mu}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right) + \rho g\end{aligned} \tag{5}$$

$$\frac{\partial}{\partial r}(\rho v) + \frac{\rho v}{r} + \frac{\partial}{\partial z}(\rho u) = 0 \tag{6}$$

where

$$\mathbf{v}\cdot\nabla = v\frac{\partial}{\partial r} + u\frac{\partial}{\partial z}, \tag{7}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}, \tag{8}$$

$$\nabla\cdot\mathbf{v} = \frac{\partial v}{\partial r} + \frac{v}{r} + \frac{\partial u}{\partial z}. \tag{9}$$

In this work, a force in the $z$-direction is imposed. As such, the force is in the flow direction and axisymmetry is maintained.

The conservation equations are coupled to a linear equation of state

$$\rho = \rho_0[1 + \beta(P - P_0)] \tag{10}$$

where $\beta$ is the isothermal compressibility, and $\rho_0$ the density at a reference pressure $P_0$, and to a linear density-dependent viscosity

$$\mu = \mu_0\left(1 + A\frac{\rho}{\rho_0}\right), \tag{11}$$

where $\mu_0$ is a constant, and $A$ a dimensionless scaling factor that we will refer to as the viscosity–density parameter or viscosity coefficient. Note that the reference viscosity is $\mu_0(1 + A)$, and that through Eq. (10), the viscosity in (11) is essentially pressure-dependent. Also note that pressure-dependence in the viscosity is controlled by both the viscosity–density parameter as well as the isothermal compressibility. For flow through an impermeable cylinder with Navier-slip at the wall, the following boundary conditions are imposed:

$$-\mu\left.\frac{\partial u}{\partial r}\right|_{r=R} = \tilde{\beta}\,u|_{r=R} \quad \text{(Navier slip at the wall)} \tag{12}$$

$$v|_{r=R} = 0 \quad \text{(impermeable wall)} \tag{13}$$

$$\left.\frac{\partial u}{\partial r}\right|_{r=0} = 0 = v|_{r=0} \quad \text{(axisymmetric flow)} \tag{14}$$

$$Q = 2\pi\int_0^R r\,(\rho u)|_{z=L}\,dr = \dot{M} \tag{15}$$

$$P|_{r=R,z=L} = P_0. \tag{16}$$