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Flow regimes in a simplified Taylor–Couette-type flow model

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ABSTRACT

In this paper we introduce a simplified variant of the well-known Taylor–Couette flow. The aim is to develop and investigate a model problem which is as simple as possible while admitting a wide range of behaviour, and which can be used for further study into stability, transition and ultimately control of flow. As opposed to models based on ordinary differential equations, this model is fully specified by a set of partial differential equations that describe the evolution of the three velocity components over two spatial dimensions, in one meridian plane between the two counter-rotating coaxial cylinders. We assume axisymmetric perturbations of the flow in a narrow gap limit of the governing equations and, considering the evolution of the flow in a narrow strip of fluid between the two cylinders, we assume periodic boundary conditions along the radial and axial directions, with special additional symmetry constraints. In the paper, we present linear stability analysis of the first bifurcation, leading to the well known Taylor vortices, and of the secondary bifurcation, which, depending on the type of symmetries imposed on the solution, can lead to wave-like solutions travelling along the axial direction. In addition, we show results of numerical simulations to highlight the wide range of flow structures that emerge, from simple uni-directional flow to chaotic motion, even with the restriction placed on the flow.

1. Introduction

The flow in the gap between two independently rotating coaxial cylinders, the Taylor–Couette (TC) flow, has been the subject of extensive research work from the early works of Taylor [1]. Because of its simple configuration, it has been a useful ground for comparison between numerical, experimental and theoretical studies, of which a detailed review is beyond the scope of this paper. Because of the large parameter space, defined by the rotation rates of the two cylinders, the ratio of the gap to the span of the cylinder and the ratio of the radii, the TC flow displays a rich phenomenology of flow regimes and bifurcations leading to turbulence, Chossat and Iooss [2], as exemplified by the experimental work of Andereck et al. [3] and the early studies of Coles [4].

When a narrow gap limit of the governing equation is considered, the equations become structurally similar to that of the rotating plane Couette flow, which has also been subject of extensive work, see for example the numerical investigations of Bech and Anderson [5,6] and the more recent study of Tsukahara et al. [7]. The flow is governed by two nondimensional parameters defining the shear rate and the rotation rate, [8]. For the plane Couette flow with system rotation, the characteristics of the first bifurcation of this system are well known, and the theoretical stability predictions of the linearised equations are in very good agreement with the experimental observation. The laminar Couette flow establishing between the two rigid boundaries shows a first supercritical bifurcation, that results in the formation of azimuthal roll cells. For this case, it is well known, as discussed by Lezius and Johnston [9], that the conditions for marginal stability for the rotating system have a complete analogy with the buoyancy driven instability in heated fluid layers. For larger values of the governing parameter, the azimuthal toroidal vortices become unstable to a class of non-axisymmetric time-dependent and time-independent disturbances, [8,10,11]. Interestingly, some of these three-dimensional nonlinear states survive to the limit of rotation going to zero, i.e. plane Couette flow, [12,13]. These tertiary states subsequently undergo a complex sequence of bifurcations, [14], leading to a large variety of flow regimes [7].

The high-Reynolds number turbulent regime as been also extensively studied, for instance with the purpose of establishing asymptotic scaling laws for the transport of angular momentum [15–19]. At high Reynolds numbers, two- as well three-dimensional

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large scale organised flow structures still persist, showing hysteretic behaviour, [20], and are known to significantly affect the turbulent transport of momentum across the gap, [21–25].

The main contribution of this paper is the discussion of a particular reduced variant of the Taylor–Couette flow. The model that we describe is derived from the original three-dimensional problem by introducing a number of simplifying assumptions, with the objective of making its mathematical analysis as simple as possible. In fact, it should be regarded mainly as a test bed for developing novel analysis methods for studies on stability, transition, turbulence and chaos rather than as a realistic model of the bifurcations and flow regimes of the full three-dimensional problem. As opposed to other TC models based on a system of ordinary differential equations, e.g. [26], the model we introduce here is given by a set of partial differential equations that describe the evolution of purely axisymmetric perturbations of the laminar flow, in a narrow gap limit of the governing equations. The main difference with earlier works, e.g. [27], is the adoption, for the unique sake of further simplification, of periodic boundary conditions along both the radial and axial directions, and the addition of special symmetries to constrain the solution of the problem. For instance, adoption of these peculiar boundary conditions allows an analytical solution of the linear and energy stability problems, whereas these require a numerical solution in the more common case of no-slip boundary conditions. The proposed TC-type flow variant adds to the class of simple flows rich in features and flow regimes, which play a special role in fluid dynamics, as for instance, the Kolmogorov flow, or the ABC flow.

Our motivation for developing and analysing this bare bone version of the TC problem lies in our interest in developing novel methods for stability and control of fluid flows, exploiting the so-called sum-of-squares-of-polynomials optimisation and semi-definite programming techniques (see [28] and reference therein for an introduction to the topic). In particular, a novel approach to nonlinear stability, i.e. stability to arbitrary finite amplitude perturbations, extending the methodology recently proposed in [29], has been applied on this system, [30]. The objective of this paper is to discuss important characteristics of this system, and in particular to present and discuss the unique flow regimes that emerge in numerical simulations of the problem. Where possible, connection between the presently-observed states and the original three-dimensional flow will be made.

In Section 2 the model flow is derived and the numerical approach developed to solve it is discussed. Section 3 contains analytical results of the linear stability analysis of first bifurcation that the flow model exhibits. In Section 4 we present numerical results of nonlinear simulations of the flow with two different sets of symmetry constraints that can be imposed on the solution, discussed in Section 2. The objective of this section is to show the characteristic flow features that the proposed flow model exhibits and to discuss the differences with respect to the full Taylor–Couette flow. In addition, the stability of the secondary flow is also analysed.

2. Problem definition and numerical methods

A sketch of the flow geometry is given in Fig. 1, where the flow between two cylinders of radii R_1 and R_2 is considered. We adopt a thin gap approximation of the problem, such that the gap ΔR is small compared to the cylinders radii. The two cylinders rotate with angular speeds ω_1 and ω_2. The frame of reference, fixed in space, is (ξ, r, θ), which are the axial, the radial and the azimuthal coordinates, respectively.

In an effort to maximally simplify the resulting model of partial differential equations, we do not consider the flow across the full width of the gap, but we study the evolution of the flow only in a layer of small thickness $\delta \ll \Delta R$ lying in the middle of the gap, at $r = R_1 + \Delta R/2$. We then introduce a rotating frame of reference $\mathbf{x} = (x, y, z)$ centred in the mid-plane between the cylinders, as illustrated in Fig. 1, which rotates around ξ with angular speed $\omega_R = (\omega_1 + \omega_2)/2$, the angular speed of the layer of fluid in the mid plane. In this frame of reference, x is the streamwise direction, y is the radial direction, while z is the axial direction, with opposite direction to ξ. As a result, a Coriolis force term appears in the equations of motion together with a constant, centrifugal, radial pressure gradient, which can be safely incorporated into the pressure variable.

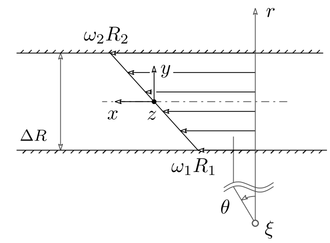

Fig. 1. Schematic representation of the physical problem, with the coordinate system adopted.

Under this frame of reference, the problem has a steady laminar solution described by a linear profile, $\mathbf{U} = (by, 0, 0)$, where the slope b is $(\omega_2 R_2 - \omega_1 R_1)/\Delta R$. The linear profile will be referred to as the Couette flow. We consider the evolution of the velocity perturbations $\mathbf{u} = (u, v, w)$ over this basic state.

Along the axial direction z we assume periodic boundary conditions. Although extensive research has established that the domain size might profoundly affect the behaviour of the solution, the preferred wavelength of the roll-cell structures, and the averaged fluxes of momentum/heat, we fix in this paper the z-period to be equal to the thickness δ for the sake of simplicity. On the inner and outer boundaries of the layer of thickness δ we assume periodic boundary conditions for the velocity perturbation vector $\mathbf{u}$. This choice is motivated mainly by the technical reason of constructing a model problem that is more amenable to mathematical analysis, so that the resulting partial differential equations can be used as a more flexible test bed for developing new analysis methods, although the connection with the flow physics of the three-dimensional problem is partially lost. For instance, the linear and energy stability problems can be solved analytically, whereas a numerical solution is required for the more common case of no-slip conditions. This property was indeed exploited in [28,30], where a novel method for nonlinear stability analysis was developed and tested on the present flow model.

A further simplifying assumption is that we consider axisymmetric perturbations, for which $\partial/\partial x = 0$. As a result, because the perturbation is independent from x, the evolution of the three-dimensional velocity vector $\mathbf{u}$ in a single two-dimensional axial–radial plane $(z - y)$ is studied. It is important to point out that this assumption has a profound impact on the flow patterns and nonlinear states that are observed in the present flow model, some of which are significantly different from those observed in three dimensional geometries. For example, some of the presently-observed flow regimes are a unique feature of the model and might be overwhelmed by other three dimensional states in a complete configuration. In fact, bifurcations leading to tertiary three-dimensional states, such as wavy vortex cells, [7], cannot arise in the present model.

Among the different possibilities of normalising the problem, we make velocities non-dimensional as $\mathbf{u}' = \mathbf{u}/\mathrm{d}U = \mathbf{u}/b\delta$, where $\mathrm{d}U = b\delta$ is the mean velocity difference between the inner and outer boundaries of the layer of fluid considered. Lengths are made non-dimensional using $\delta/2\pi$, such that $\mathbf{x}' = 2\pi\mathbf{x}/\delta$, and the