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Triad resonance between a surface-gravity wave and two high frequency hydro-acoustic waves

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ABSTRACT

A new class of resonant triad in the family of gravity–acoustic waves has been found. I show that a hydro-acoustic wave interacting with a surface-gravity wave may generate a second hydro-acoustic wave. Interestingly, the two acoustic waves propagate in the same direction with similar wavelengths, that are almost double of that of the gravity wave. The evolution of the wave triad amplitudes is periodic and it is derived analytically, in terms of *Jacobian elliptic* functions and *elliptic integrals*. The physical importance of this type of triad interactions is the modulation of pertinent hydro-acoustic signals, leading to inaccurate signal perceptions.

1. Introduction

Resonance interactions of waves play a prominent role in energy share among the different wave types involved. Such interactions may significantly contribute, among others, to the evolution of the ocean energy spectrum by exchanging energy between surface-gravity waves [1,2]; surface and internal gravity waves [3–5]; or even surface and compression-type waves [6–8], that can transfer energy from the upper ocean through the whole water column reaching down to the seafloor [9–11].

A resonant triad occurs among a triplet of waves, usually involving interaction of nonlinear terms of second order perturbed equations. Until recently, it has been believed that in a homogeneous fluid a resonant triad is possible only when tension forces are included, or at the limit of a shallow water [12]. Moreover, [9] argued that when the compressibility of water is considered, no resonant triads can occur within the family of gravity–acoustic waves. However, [7] proved that, under some circumstances, resonant triads comprising two opposing surface-gravity waves of similar periods (though not identical) and a much longer acoustic–gravity[1] wave, of almost double the frequency, exist.

In this paper I report on a new resonant triad involving a surface-gravity wave and two hydro-acoustic waves of almost double the length. Since the lengths of the gravity and acoustic waves are comparable, the present resonance is relevant to hydro-acoustic waves of relatively high frequency. This resonance is a second of its kind in the family of gravity–acoustic waves, and it has a significantly different characteristics compared to other resonant triads. Here, even though the interaction of gravity and acoustic modes does not concern short and long waves, the corresponding frequencies are disparate.

2. Background

2.1. Governing equations

Consider a two dimensional Cartesian coordinate system (x, z) with the origin in the undisturbed free surface, and the z-axis vertically upwards. Let $z = \eta$ be the equation of the free surface, and $z = -h$ the equation of the rigid flat bottom. Assume that the density is a function of pressure alone, the viscosity is negligible, and the velocity $\mathbf{u}$ is irrotational, so that $\mathbf{u} = \nabla\varphi$. Approximate to quadratic terms, the equations of motion can then be integrated to obtain the field equation [6]

$$\varphi_{tt} - c^2\nabla^2\varphi + g\varphi_z = -2\varphi_x\varphi_{xt} - 2\varphi_z\varphi_{zt} \quad (-h \le z \le 0), \tag{1}$$

where c is the speed of sound in the fluid, g is the acceleration due to gravity, and t is the time. The boundary condition at the bottom is

$$\varphi_z = 0 \quad (z = -h). \tag{2}$$

From the continuity equation we know that $\mathrm{D}\rho/\mathrm{D}t - \rho\nabla^2\varphi = 0$, where $(\mathrm{D}/\mathrm{D}t)$ is the differentiation following motion, and ρ

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[1] Acoustic–gravity wave is a very long hydro-acoustic wave that is affected by the force of gravity.

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is the fluid density. Since the flow is barotropic and a particle at the free surface forever remains there, where the pressure is atmospheric, the kinematic boundary condition is reduced to $\nabla^2\varphi = 0$ (e.g. see [6]). On the other hand, the dynamic free surface boundary condition, $\varphi_t+(\varphi_x^2+\varphi_z^2)/2+g\eta = 0$, is obtained from the equation of motion. Expanding the kinematic condition in a Taylor series around $z = 0$ and making use of the dynamic conditions, [7] obtained the combined condition at the free surface

$$g\nabla^2\varphi = \varphi_t\varphi_{xxz} + \varphi_t\varphi_{zzz} \quad (z = 0). \tag{3}$$

An alternative formulation of the free surface boundary condition can be obtained after some simple manipulation of Eqs. (1) and (3) (see [7])

$$\varphi_{tt} + g\varphi_z = -2\varphi_x\varphi_{xt} - 2\varphi_z\varphi_{zt} + g^{-1}\varphi_t\varphi_{ttz} + \varphi_t\varphi_{zz} \quad (z = 0), \tag{4}$$

where, to quadratic order, (1) together with (3), are equivalent to (1) with (4).

2.2. Linear solution

Neglecting nonlinear terms in (1) and (3), and for practical purposes neglecting the gravity term in (1) and seeking a progressive-wave solution with frequency ω, the linear solution is obtained [13]

$$\varphi = \frac{giA}{2\omega}\frac{\cosh\left[\mu\,(h+z)\right]}{\cosh(\mu h)}e^{i(kx-\omega t)} + \text{c.c.}, \tag{5}$$

where c.c. denotes complex conjugates, $k^2 = \mu^2 + \omega^2/c^2$, and the dispersion relation is given by

$$\omega^2 = g\mu\tanh(\mu h). \tag{6}$$

The leading root in the dispersion relation (6) is real with a corresponding real wavenumber resembling the surface-gravity mode. On the other hand, the remaining infinity of roots are all imaginary. At any prescribed frequency ω, and water depth $h > h_{cr} \equiv \pi c/2\omega$ there is at least one propagation mode, with imaginary root but real wavenumber. Such modes are referred to as hydro-acoustic. All remaining modes having imaginary wavenumbers and are known as evanescent modes. Thus, for the acoustic modes, provided that $\lambda = i\mu$ is real, we can exclusively write

$$\omega^2 = -g\lambda\tan(\lambda h) \tag{7}$$

and

$$k^2 = \frac{\omega^2}{c^2} - \lambda^2 > 0. \tag{8}$$

Note that for a gravity wave of frequency $\omega = \sigma$, travelling in deep water, $\mu \simeq k$, the dispersion relation reduces to

$$\sigma^2 = gk. \tag{9}$$

3. Resonant triads

Concerning a triad involving two acoustic modes of frequencies ω_1 and ω_2, and wavenumbers q_1 and q_2, and a single gravity mode of frequency σ and wavenumber k, we are seeking to satisfy the resonance conditions

$$\sigma = \omega_1 - \omega_2, \qquad k = q_1 + q_2, \tag{10}$$

and the dispersion relations (7) and (9) for the acoustic and gravity modes, respectively. From the gravity mode dispersion relation (9) we can combine the resonance conditions,

$$(\omega_1 - \omega_2)^2 = g\,(q_1 + q_2)\,. \tag{11}$$

On the other hand, for the hydro-acoustic waves it is known that $\lambda h = \pi/2+\Delta$, for the first mode, where $\Delta \ll 1$. Upon substitution in (7), the dispersion relation of the first acoustic mode reduces to

$$\omega^2 = \frac{g\lambda}{\lambda h - \pi/2}, \tag{12}$$

to leading order in Δ. Isolating λ and substituting in (8) and (11) we obtain a relation between ω_1 and ω_2,

$$(\omega_1-\omega_2)^2 = g\left(\sqrt{\frac{\omega_1^2}{c^2} - \frac{\omega_1^4\pi^2/4}{\left(\omega_1^2 h - g\right)^2}} + \sqrt{\frac{\omega_2^2}{c^2} - \frac{\omega_2^4\pi^2/4}{\left(\omega_2^2 h - g\right)^2}}\right). \tag{13}$$

By requiring wavenumbers to be real, it is easy to show that the acoustic cut-off frequency is $\omega_{cr} \simeq \pi c/2h$, which corresponds to a Longuet-Higgins resonance (see [8,6]). For any $\omega_2 > \omega_{cr}$ one can always find ω_1, from (13), and σ, from (11), that satisfy the dispersion relations and resonance conditions.

4. Amplitude evolution equations

For the resonant triad case, we assume that the complex amplitudes of the gravity mode $S(\tau)$, and the two acoustic modes $A_1(\tau)$, and $A_2(\tau)$, are all slow variables in time τ. The first order velocity potential of the triad is given by

$$\phi^{(1)} = S(\tau)e^{kz}e^{i(kx-\sigma t)} + A_1(\tau)\cos[\lambda_1(z+h)]e^{-i(q_1x+\omega_1 t)} + A_2(\tau)\cos[\lambda_2(z+h)]e^{i[(k-q_1)x-(\sigma+\omega_1)t]} + \text{c.c.} \tag{14}$$

The governing equations for the second order potential are the field equation

$$\phi_{tt}^{(2)} - c^2\nabla^2\phi^{(2)} + g\phi_z^{(2)} = -2\phi_{\tau t}^{(1)} - 2\phi_x^{(1)}\phi_{xt}^{(1)} - 2\phi_z^{(1)}\phi_{zt}^{(1)} \quad (-h < z < 0), \tag{15}$$

the bottom boundary condition

$$\phi_z^{(2)} = 0 \quad (z = -h), \tag{16}$$

and the combined surface condition

$$\phi_{tt}^{(2)} + g\phi_z^{(2)} = -2\phi_{\tau t}^{(1)} - 2\phi_x^{(1)}\phi_{xt}^{(1)} + 2g^{-1}\phi_{tt}^{(1)}\phi_{zt}^{(1)} + 2g^{-1}\phi_{ttt}^{(1)}\phi_z^{(1)} - g^{-2}\phi_{tttt}^{(1)}\phi_t^{(1)} \quad (z = 0). \tag{17}$$

We define the second order potential with amplitudes changing slowly in time

$$\phi^{(2)} = F_S(z,\tau)e^{i(kx-\sigma t)} + F_{A_1}(z,\tau)e^{-i(q_1x+\omega_1 t)} + F_{A_2}(z,\tau)e^{i[(k-q_1)x-(\sigma+\omega_1)t]} + \text{c.c.} \tag{18}$$

4.1. Derivation of the evolution equations

Substituting (18) in the field equation, we write for the gravity wave

$$F_{S,zz} - k^2F_S = -\frac{2i\sigma}{c^2}e^{kz}S_\tau + \frac{i\sigma}{c^2}\Big\{\left(-q_1^2 - \lambda_1\lambda_2 + q_1k\right) \times \cos\left[(\lambda_1-\lambda_2)(z+h)\right] + \left(-q_1^2 + \lambda_1\lambda_2 + q_1k\right) \times \cos\left[(\lambda_1+\lambda_2)(z+h)\right]\Big\}A_1^*A_2. \tag{19}$$