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The complexity of decision problems about equilibria in two-player Boolean games

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ABSTRACT

Boolean games allow us to succinctly represent strategic games with binary payoffs in the case where the players' preferences have a structure readily expressible in propositional logic. Since their introduction, the computational aspects of Boolean games have been of interest to the multiagent community, but so far the focus has been exclusively on pure strategy equilibria. In this paper we consider the complexity of problems involving mixed strategy equilibria, such as the existence of an equilibrium satisfying a given payoff constraint. The results are obtained by the observation that a mixed strategy can hold enough information to encode the computation history of an exponential time Turing machine.

1. Introduction

The ideas of game theory find fertile ground in the multiagent framework. Any analysis of a system composed of self-interested agents, whether human or otherwise, cannot ignore the fact that an agent's behaviour is going to be influences by the choices of the agents around him, and in turn his choice of action will necessitate that the other agents respond to him.

Unfortunately games tend to get very large very quickly—the standard normal form representation is of size $O(S^n)$ for a game with n players possessing S strategies each. This has motivated the study of succinct game representations.

The Boolean game [7] is a natural and expressive game representation that achieves succinctness by interpreting a player's strategy as an assignment of truth values and his preferences as a formula of propositional logic. Given n players with formulae of length k, this gives a representation of size $O(nk)$. The dependency on the number of players is reduced to linear, and while no a priori upper bound can be given on k, non-trivial games can be represented with formulae of size logarithmic in the number of strategies.

There is a wide literature on complexity considerations for Boolean games and related extensions, but, with the exception of a preliminary version of this paper [8], these have said next to nothing about the complexity of mixed equilibria.

1.1. Our contribution

We demonstrate that the following problems are NEXP-complete for two-player Boolean games:

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- Given a Boolean game G and a vector of payoffs $\boldsymbol{v}$, determine whether G has an equilibrium where Player i gets at least $\boldsymbol{v}[i]$ utility.
- Given a Boolean game G and a formula of propositional logic φ, determine whether G has an equilibrium where φ is satisfied with a probability of 1.

And that the following are coNEXP-complete for two-player Boolean games:

- Given a Boolean game G and a formula of propositional logic φ, determine whether φ is satisfied with a probability of 1 in every equilibrium of G.
- Given a Boolean game G, determine whether G has a unique equilibrium.

1.2. Related work

Boolean games were first introduced by Harrenstein et al. [7] as two-player, zero-sum games induced by a formula of propositional logic. The multiplayer definition used today is due to [1]. Algorithmic studies (e.g. [2,1,3]) focused on the complexity of problems involving pure equilibria, where it was shown that determining whether a pure strategy equilibrium exists is Σ_2^p-complete, but the complexity can be lower if the players' goal formulae are restricted to a tractable fragment of propositional logic.

Boolean games can be seen to be a special case of circuit games [4,10]. In a circuit game a player is equipped with a Boolean circuit, control of the input gates is distributed among the players, and the output of the circuit is an integer representing the player's payoff. A Boolean game, then, is a circuit game where the output of the circuit is limited to 0 or 1 and the circuit has fan-out 1. As a consequence of this, a complexity result for circuit games is an upper bound for the complexity of the same problem for Boolean games, and a result for Boolean games is a lower bound for circuit games. In particular, this means the result of [10] that ∃GuaranteeNash is NEXP-complete for circuit games does not imply the result in this paper.

2. Preliminaries

2.1. Strategic and Boolean games

We start with standard notions of game theory:

Definition 2.1. A *strategic game* is a triple $(N, \{S_1, \dots, S_n\}, \{u_1, \dots, u_n\})$. N is a finite set of *players*, of cardinality n. S_i is a finite set of Player i's *pure strategies*. An n-tuple of pure strategies, i.e. a member of $\mathcal{S} = S_1 \times \cdots \times S_n$ is called a *pure-strategy profile*. The function $u_i : \mathcal{S} \to \mathbb{R}$ is Player i's *utility function*.

A strategic game is called *zero-sum* just if there exists a $c \in \mathbb{R}$ such that for every $\boldsymbol{s} \in \mathcal{S}$:

$$\sum_{i \in N} u_i(\boldsymbol{s}) = c. \quad \blacksquare$$

Definition 2.2. A pure-strategy profile $\boldsymbol{s}$ is called a *pure-strategy equilibrium* just if for each $i \in N$, for all $s' \in S_i$:

$$u_i(\boldsymbol{s}) \geq u_i(\boldsymbol{s_{-i}}(s')),$$

where $\boldsymbol{s_{-i}}(s')$ denotes the profile obtained by replacing Player i's strategy in $\boldsymbol{s}$ with s'. $\blacksquare$

Definition 2.3. Let $\mathcal{P}(S_i)$ denote the space of probability distributions over S_i. A *mixed strategy* for Player i is a member of $\mathcal{P}(S_i)$. The weight assigned to a pure strategy s by a mixed strategy σ, or $P(s \mid \sigma)$, is called the *strategy weight* of s.

An n-tuple of mixed strategies, $\boldsymbol{\sigma} \in \mathcal{P}(\mathcal{S})$, is called a *mixed-strategy profile*. We extend Player i's utility function to the space of mixed-strategy profiles on the principle of expected utility. That is:

$$u_i(\boldsymbol{\sigma}) = \sum_{\boldsymbol{s} \in \mathcal{S}} u_i(\boldsymbol{s}) P(\boldsymbol{s} \mid \boldsymbol{\sigma}).$$

A mixed-strategy profile is called a *mixed-strategy equilibrium* just if for all $s' \in S_i$:

$$u_i(\boldsymbol{\sigma}) \geq u_i(\boldsymbol{\sigma_{-i}}(s')). \quad \blacksquare$$

We have need of two classical results about games.

Theorem 2.4 *([11]). In every two-player zero-sum game there exists a v such that Player One gets a utility of v in every equilibrium. This v is called the* value *of the game.*