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Extending fuzzy logics with many hedges

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Abstract

Fuzzy logic aims at modeling logical reasoning with vague or imprecise statements, which may contain linguistic hedges. In fact, many hedges, e.g., *very*, *highly*, *rather*, and *slightly*, can be used simultaneously to express different levels of emphasis. Moreover, each hedge might have a dual one, e.g., *slightly* can be seen as a dual hedge of *very*. Thus, it is necessary to extend systems of fuzzy logic with multiple hedges. This work proposes two axiomatizations for multiple hedges as an expansion of a core fuzzy logic. In one axiomatization, hedges do not have any dual one while in the other, each hedge can have its own dual one. It is shown that the proposed logics not only cover a large class of hedge functions but also have all completeness properties as the underlying logic w.r.t. the class of their chains as well as distinguished subclasses of their chains, including standard completeness. The axiomatizations are also extended to the first-order level. Furthermore, we present a method to build linguistic fuzzy logics based on the axiomatizations and a hedge algebra, whose corresponding algebras use a linguistic truth domain taken from the hedge algebra, for representing and reasoning with linguistically-expressed human knowledge, where truth values of vague sentences are given in linguistic terms.

Keywords: Mathematical fuzzy logic; t-Norm based logics; Axiomatization; Hedge; Completeness; Hedge algebra

1. Introduction

Previous works on adding hedges to logical systems of mathematical fuzzy logic (MFL) [9] include those by Hájek [18], Vychodil [37], and Esteva et al. [14]. In MFL, hedges are called *truth-stressing* or *truth-depressing* depending on whether they strengthen or weaken the meaning of the proposition. The intuitive interpretation of a truth-stressing (resp., truth-depressing) hedge on a chain of truth values is a subdiagonal (resp., superdiagonal) non-decreasing function preserving 0 and 1. Such functions are called *hedge functions*. Hájek [18] introduces an axiomatization of a truth-stressing hedge vt as an expansion of Basic Logic (BL) [17], and the resulting logic is called BL_{vt}. Vychodil [37] extends BL_{vt} to a logic $\mathrm{BL}_{vt,st}$ with a truth-depressing hedge *st* dual to *vt*. The logics are shown to be algebraizable and enjoy completeness w.r.t. the classes of their chains, but are not proved to enjoy standard completeness in

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general. Moreover, Hájek and Vychodil's axiomatizations do not cover a large class of hedge functions [14]. Hence, Esteva et al. [14] propose weaker axiomatizations over any core fuzzy logic for a truth-stressing hedge or/and a truth-depressing one, which do not impose any more constraints on hedge functions, and the axiomatizations are proved to enjoy standard completeness.

This work proposes two axiomatizations over any propositional core fuzzy logic for multiple truth-stressing and truth-depressing hedges, one for non-dual hedges and the other for dual ones. The axiomatizations not only cover a large class of hedge functions but also have all completeness properties of the underlying core fuzzy logic w.r.t. the class of their chains and distinguished subclasses of their chains, including standard completeness. The axiomatizations are also extended to the first-order level. Moreover, we show how to build linguistic fuzzy logics based on the axiomatizations and hedge algebras [32,33] for representing and reasoning with linguistically-expressed human knowledge.

The remainder of the paper is organized as follows. Section 2 gives an overview of MFL, previous axiomatizations for hedges, linguistic truth domains and operations on them. Section 3 presents an axiomatization for multiple non-dual hedges while Section 4 provides an axiomatization for multiple dual ones. Section 5 shows how to build linguistic fuzzy logics. Section 6 extends the axiomatizations to the first-order level. Section 7 discuses related work. Finally, Section 8 concludes the paper.

2. Preliminaries

2.1. Preliminaries on mathematical fuzzy logic

Let L be a logic in a language $\mathcal{L}$, a set of connectives with finite arity. A truth constant $\overline{r}$ is a special formula whose truth value under each evaluation is r. Formulae are built from propositional variables and truth constants using connectives in $\mathcal{L}$. Each evaluation e of propositional variables by truth values uniquely extends to an evaluation $e(\varphi)$ of all formulae φ using truth functions of connectives, which are operations of an algebra used to interpret the language. A formula φ is called an *1-tautology* if $e(\varphi) = 1$ for every evaluation e. A number of 1-tautology formulae are taken as *axioms* of the logic. A *theory* is a set of formulae called *special axioms*. An evaluation e is called a *model* of a theory T if $e(\varphi) = 1$ for all φ in T. A *proof* in a theory T is a sequence $\varphi_1, \ldots, \varphi_n$ of formulae such that each element of the sequence is either an axiom of the logic or an element of T or follows from some preceding elements of the sequence using the deduction rule(s) of the logic. The last member φ of a proof in T is called a *provable* formula, denoted $T \vdash_L \varphi$. If $T = \emptyset$, it is said that φ is *provable* in the logic [17,9].

It is said that L is a *Rasiowa-implicative logic* [35,10] if there is a binary (either primitive or definable by a formula) connective $\to$ in its language such that:

(R) $\vdash_L \varphi \to \varphi$, (MP) $\varphi, \varphi \to \psi \vdash_L \psi$,
(W) $\varphi \vdash_L \psi \to \varphi$, (T) $\varphi \to \psi, \psi \to \chi \vdash_L \varphi \to \chi$,
(sCng) $\varphi \to \psi, \psi \to \varphi \vdash_L c(\chi_1, \ldots \chi_i, \varphi, \ldots, \chi_n) \to c(\chi_1, \ldots \chi_i, \psi, \ldots, \chi_n)$
for each n-ary $c \in \mathcal{L}$ and each $i < n$.

Every finitary Rasiowa-implicative logic L is algebraizable and its equivalent algebraic semantics, a class of L-algebras, is a quasivariety [5], denoted $\mathbb{L}$. The algebraic semantics enjoys the following completeness.

Theorem 1 *(Strong completeness). [9] For every set $T \cup \{\varphi\}$ of formulae, $T \vdash_L \varphi$ iff for every $\mathbf{A} \in \mathbb{L}$ and every $\mathbf{A}$-model e of T, $e(\varphi) = 1$.*

Every L-algebra $\mathbf{A}$ is endowed with a *preorder* relation $\leq$ by setting, for every $a, b \in A$, $a \leq b$ iff $a \Rightarrow b = 1$, where $\Rightarrow$ is the truth function of $\to$. $\mathbf{A}$ is called an L-*chain* if $\leq$ is a total order, i.e., A is linearly ordered. L is called a *semilinear* logic iff it is strongly complete w.r.t. the class of L-chains or, equivalently, if every L-algebra is representable as subdirect product of L-chains [3,10].

In the literature, most logical systems referred to as *fuzzy logics* are, indeed, a finitary Rasiowa-implicative semilinear logic. They belong to a large class of systems which are axiomatic expansions of MTL (*monoidal t-norm based logic*) satisfying (sCng) for any possible new connective [13]. Such systems are called *core fuzzy logics*. Well-known

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