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Meager projections in orthocomplete homogeneous effect algebras

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Abstract

We define meager projections on homogeneous effect algebras and discuss their properties. As an application, we prove that, if E is an orthocomplete homogeneous effect algebra such that $\mathrm{S}(E)$ is lattice-ordered, then E is a lattice effect algebra, which gives an affirmative answer to an open problem stated by Mesiar and Stupňanová.

Keywords: Effect algebra; Homogeneous; Sharp element; Meager element; Meager projection

1. Introduction

In the nineties of the twentieth century, two equivalent quantum structures, D-posets [13] and effect algebras [3] were extensively studied, which were considered as "unsharp" generalizations of the structures which arise in quantum mechanics (orthomodular lattices, orthomodular posets, orthoalgebras) incorporating some fuzzy logics (MV-algebras).

Effect algebras are partially ordered by a natural way, and if they are lattices then they are called lattice effect algebras. The set of all sharp elements of a lattice effect algebra E is an orthomodular lattice, being a sub-effect algebra and a sublattice of E [7]. In [8], a new class of effect algebras, called homogeneous effect algebras, was introduced. The class of homogeneous effect algebras includes orthoalgebras, effect algebras satisfying the Riesz decomposition property and lattice effect algebras. The set of all sharp elements of a homogeneous effect algebra is its sub-effect algebra [8].

Below we present Problem 10.1 from [14], which was posed by Jenča during The Twelfth International Conference on Fuzzy Set Theory and Applications 2014.

Problem 1.1. ([14]) Prove or disprove: If E is an orthocomplete homogeneous effect algebra such that $\mathrm{S}(E)$ is lattice-ordered, then E is a lattice effect algebra.

In this paper, we give an affirmative answer to this problem using the properties related to meager projections on orthocomplete homogeneous effect algebras.

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2. Preliminaries and basic facts

Effect algebras have been introduced by Foulis and Bennett to study the foundations of quantum mechanics [3]. An *effect algebra* is a partial algebra $(E; \oplus, 0, 1)$ with a binary partial operation $\oplus$ and two nullary operations 0, 1, satisfying the following conditions for any $x, y, z \in E$:

(E1) if $x \oplus y$ is defined, then $y \oplus x$ is defined and $x \oplus y = y \oplus x$,
(E2) if $x \oplus y$ and $(x \oplus y) \oplus z$ are defined, then $y \oplus z$ and $x \oplus (y \oplus z)$ are defined and $(x \oplus y) \oplus z = x \oplus (y \oplus z)$,
(E3) for every $x \in E$ there is a unique $x' \in E$ such that $x \oplus x'$ exists and $x \oplus x' = 1$,
(E4) if $x \oplus 1$ is defined, then $x = 0$.

We often denote the effect algebra $(E; \oplus, 0, 1)$ briefly by E. For any element x in an effect algebra, the element x' in (E3) is equal to $1 \ominus x$ and called the *orthosupplement* of x. The unary operation $' : E \to E$ is involutive and order reversing. In every effect algebra we can define the partial operation $\ominus$ and the partial order $\leq$ by putting $x \leq y$ and $y \ominus x = z$ if and only if $x \oplus z$ is defined and $x \oplus z = y$. The partial operations $\oplus$ and $\ominus$ are connected by the rules $a \oplus b = (a' \ominus b)'$ and $a \ominus b = (a' \oplus b)'$.

If E is an effect algebra such that $(E; \leq)$ is lattice-ordered, then we say that E is a *lattice effect algebra*. An *orthogonality* relation on E is defined by $x \perp y$ if and only if $x \oplus y$ is defined. It can be proved that $x \perp y$ if and only if $x \leq y'$, if and only if $y \leq x'$. For convenience, when we write $x \oplus y$, we mean that $x \oplus y$ is defined. If x, y in E are such that $x \leq y'$ and $x \vee y$ exists, then $x \wedge y$ exists and $x \oplus y = (x \vee y) \oplus (x \wedge y)$ [5]. In particular, $x \vee y \leq x \oplus y$ and the equality is valid if and only if $x \wedge y = 0$ [19]. We refer to [1] for more information on effect algebras and related topics.

Elements $x, y \in E$ are *disjoint* if and only if $x \wedge y = 0$. An effect algebra E satisfying $x \perp x \Rightarrow x = 0$, equivalently, $x \wedge x' = 0$ for all $x \in E$, is an orthoalgebra [1,2]. An effect algebra E is an orthomodular poset if and only if for all $x, y \in E$, $x \perp y \Rightarrow x \wedge y = 0$. We note that conversely, an orthomodular poset can be made an effect algebra by defining the $\oplus$-operation as the supremum of orthogonal elements. Clearly, an orthomodular poset is an orthoalgebra [17]. A lattice effect algebra is an MV-effect algebra if and only if $x \wedge y = 0 \Rightarrow x \perp y$.

For any $x \in E$, let $E[0, x] = \{y \mid y \leq x \text{ and } y \in E\}$. The interval $E[0, x]$ can be made an effect algebra if we restrict the operation $\oplus$ letting x act as the unit element. Thus, for $y, z \in E[0, x]$, $y \oplus_x z$ is defined and $y \oplus_x z := y \oplus z$ if and only if $y \oplus z$ exists in E and $y \oplus z \leq x$. Similarly, for $y, z \in E[0, x]$, $y \wedge_x z$ is defined and $y \wedge_x z := y \wedge z$ if and only if $y \wedge z$ exists in E.

Recall that $Q \subseteq E$ is called a *sub-effect algebra* of E if and only if

(i) $1 \in Q$,
(ii) if out of elements $x, y, z \in E$ with $x \oplus y = z$ two are in Q, then $x, y, z \in Q$.

Note that if Q is a sub-effect algebra of E then Q with inherited operation $\oplus$ is an effect algebra in its own right.

An element $s \in E$ is called a *sharp* element of an effect algebra E if $s \wedge s' = 0$ [6]. The set of all sharp elements of E is denoted by $\mathrm{S}(E)$. It has been shown that in every lattice effect algebra E the set $\mathrm{S}(E)$ is an orthomodular lattice, being a sub-effect algebra and a sublattice of E [7]. An element $p \in E$ is *principal* if for $x, y \in E$, $x \perp y$ and $x, y \leq p$ imply $x \oplus y \leq p$ [5]. An element $z \in E$ is *central* if for every $x \in E$, $(x \wedge z) \oplus (x \wedge z') = x$ [5]. The center $\mathrm{C}(E)$ is the set of all central elements in E. In every effect algebra, every principal element is sharp and every central element is principal.

An effect algebra satisfies the *Riesz decomposition property* (RDP) if, for $x_1, x_2, y_1, y_2 \in E$, $x_1 \oplus x_2 = y_1 \oplus y_2$ implies the existence of $z_{ij} \in E$, such that $x_i = z_{i1} \oplus z_{i2}$ and $y_j = z_{1j} \oplus z_{2j}$ for all $i, j \in \{1, 2\}$ [1]. Alternatively, E has the Riesz decomposition property if and only if $x \leq y_1 \oplus y_2$ implies there exist $x_1, x_2 \in E$, such that $x = x_1 \oplus x_2$, $x_1 \leq y_1$ and $x_2 \leq y_2$. If E is an effect algebra E satisfying the Riesz decomposition property and $x \in E$, then x is central if and only if x is principal, if and only if x is sharp [8]. An effect algebra is an MV-effect algebra if and only if it is lattice ordered and has the RDP [1].

We say that a finite system $F = (x_k)_{k=1}^n$ of not necessarily different elements of an effect algebra E is *orthogonal* if $x_1 \oplus x_2 \oplus \cdots \oplus x_n$ (written $\bigoplus_{k=1}^{n} x_k$ or $\bigoplus F$) exists in E. Here we define $x_1 \oplus x_2 \oplus \cdots \oplus x_n = (x_1 \oplus x_2 \oplus \cdots \oplus x_{n-1}) \oplus x_n$

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