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Short communication

A lattice-theoretical characterization of the family of cut sets of interval-valued fuzzy sets

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Abstract

In this paper the answer to a problem posed in [19] is given using known and some newly introduced types of lattices and lattice constructions. The main result is that the collection of cuts of an interval-valued fuzzy set forms a complete finitely spatial meet-between planar lattice.

Keywords: Interval-valued fuzzy sets; Lattice representations; Cuts

1. Introduction

Among generalizations of fuzzy sets, some of the most applicable are interval-valued fuzzy sets having as the codomain of the membership function the set of all closed subintervals of the unit interval [0, 1]. Since interval-valued fuzzy set can express imprecision and uncertainty, i.e. incompleteness of information, interval-valued fuzzy sets are studied extensively in recent years (e.g. in [4,10,11,19,37]). Interval-valued fuzzy sets have been used in approximate reasoning [3,9], decision making [16], image processing and pattern recognition [5,7,34], recently in mathematical morphology [25,27,30], and other applications. Interval-valued fuzzy sets are mathematically equivalent to closed interval type-2 fuzzy sets [8,26], and similarly, interval-valued fuzzy sets are syntactically equivalent with Atanassov's intuitionistic fuzzy sets and vague sets [12]. Thus, the list of potential applications of interval-valued fuzzy sets can be extended with those which are connected with Atanassov's intuitionistic fuzzy sets, vague sets and interval type-2 fuzzy sets.

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In the mentioned generalizations as well as in the classical theory, the concept of cut set (p- or α-cut set) is one of the basic and most important concepts. It is well known that a fuzzy set can be uniquely represented by the family of all its cuts. Many researchers have recognized the importance of cut sets and need for the representation of fuzzy structures by cut sets (see e.g. [2,18,25,32,33], and references given there). So, we can use a family of sets to represent fuzzy sets and their generalizations, and we want to know under which conditions it is possible. This problem was firstly posed and solved by Negoita and Ralescu [28] for fuzzy sets, and further broadened by other authors, see e.g. [22] and many other papers. Problem of the reconstruction of fuzzy topological or a fuzzy neighborhood space from an a priori given family of level-topologies was solved by Wuyts [35,36]. Recently, a representation theorem of L-subsets on complete residuated lattice and representation theorem of intersection-preserving L-families on complete residuated lattice were obtained by Han and Fang [21]. Also related to this topic is level-based representation, which is new and "an alternative tool for representing and operating with fuzziness" [29].

For lattice valued fuzzy sets, a similar problem was formulated in the following way:

Let L be a fixed complete lattice. Characterize under which conditions $\mathcal{F} \subseteq \mathcal{P}(X)$ is a collection of cut sets of a lattice valued fuzzy set $\mu : X \to L$.

This problem was solved in [18], and in a general form for poset-valued fuzzy sets in [31].

As Kerre [24] pointed out, the question about conditions under which a fuzzy structure can be reconstructed from the knowledge of its α-levels is relevant in every fuzzification of a crisp notion and in every generalization of fuzzy sets.

Therefore, it is natural to consider a similar problem for interval-valued fuzzy sets. Characterization of a collection of sets $\mathcal{F} \subseteq \mathcal{P}(X)$, which can be a collection of cut sets of an interval-valued fuzzy set $\mu : X \to L$, is given only in the finite case [19]. Although the interval-valued fuzzy sets can be viewed as lattice valued fuzzy sets, a more specific characterization is given here for family $\mathcal{F} \subseteq \mathcal{P}(X)$ to be a collection of cut sets of some interval-valued fuzzy set.

2. Preliminaries

We recall notions from lattice theory that are used in the text. The reader can find more details about topics connected with lattices e.g. in [1] or in [20].

A *complete lattice* L is a partially ordered set (poset $(L, \leq)$) in which every subset has a least upper bound and a greatest lower bound. A complete lattice has the top and the bottom element which are usually denoted by 1 and 0, respectively. The lattice is a poset $(L, \leq)$ in which every two-element set has a supremum (denoted by $\vee$) and an infimum (denoted by $\wedge$). Therefore, the lattice $(L, \leq)$ is often denoted by $(L, \vee, \wedge)$.

The dual relation to relation $\leq$ on L, $\geq$ is defined by $x \geq y$ if and only if $y \leq x$.

If a poset $(L, \leq)$ is a complete lattice, then also the poset $(L, \geq)$ is a complete lattice and it is called the dual complete lattice.

If $(L, \vee, \wedge)$ is a lattice, then the dual lattice is $(L, \wedge, \vee)$.

When a condition which is satisfied in a poset (or a lattice) is also satisfied in the dual poset (or the dual lattice), we say that the condition is dually satisfied. In the dual case an infimum $\wedge$ is exchanged with the supremum $\vee$ and vice versa.

Let L be a lattice and $L_1 \subseteq L$. We call L_1 a *complete meet-sublattice* of L if L and L_1 are complete lattices and all infima in L and L_1 coincide.

An element a of a lattice L is *join-irreducible* if it is not the bottom element and from $a = x \vee y$ for $x, y \in L$ it follows that $a = x$ or $a = y$. Meet-irreducible elements are defined dually.

We denote by J_L the set of all join-irreducible elements in a lattice L. Set J_L is ordered by the order $\leq$ inherited from L and we say that $J_L = (J_L, \leq)$ is the poset of join-irreducibles of L.

The symbol $\downarrow a = \{x \in L \mid x \leq a\}$ denotes the *principal ideal* generated by an element a.

Let $(P, \leq)$ and $(Q, \leq)$ be posets (lattices). An injection $f : P \to Q$ is an *order-embedding* from P into Q if $x \leq y$ if and only if $f(x) \leq f(y)$, for all $x, y \in P$.

We say that an order-embedding which preserves all infima is a *meet-embedding* and dually, an order-embedding which preserves all suprema is a *join-embedding*.

Let $(L, \leq)$ be a lattice. Then $M \subseteq L$ is a *chain* if any two elements of M are comparable and M is an *antichain* if any two elements of M are incomparable.

The *width* of L, denoted by $w(L)$, is the size of the largest antichain in L.