ARTICLE IN PRESS
FSS:7180

Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems ••• (••••) •••–•••

FUZZY sets and systems

www.elsevier.com/locate/fss

Fuzzy semistar operations on overrings

Olivier A. Heubo-Kwegna

Department of Mathematical Sciences, Saginaw Valley State University, 7400 Bay Road, University Center, MI 48710-0001, USA

Received 5 January 2016; received in revised form 28 February 2017; accepted 2 March 2017

Abstract

In this paper, we reinvestigate the notion of fuzzy semistar operation introduced in [5]. We show how a fuzzy semistar operation on an integral domain R induces canonically a fuzzy semistar operation on an overring T of R and conversely how a fuzzy semistar operation on T induces canonically a semistar operation on R. As an application, new characterizations of Prüfer domains and complete description of the set of all fuzzy semistar operations of finite character on Prüfer domains are obtained. We also characterize conducive domains.

Keywords: Fractionary fuzzy ideal; Fuzzy star operation; Fuzzy semistar operation; Prüfer domain

1. Introduction

The translation of multiplicative ideal theory results into the language of fuzzy theory has been of great interest to the fuzzy commutative algebra community since the introduction of the notion of fractionary fuzzy ideal and the notion of invertible fractionary fuzzy ideal by Lee and Mordeson in [8,9]. This introduction has led to the fuzzification of one of the main results in multiplicative ideal theory, that is, the characterization of Dedekind domains in terms of the invertibility of certain fractionary fuzzy ideals. In the same direction, the notion of fuzzy star operation has been introduced in [7] to provide more characterizations of integral domains in terms of fuzzy star operations. Just very recently, the more flexible notions of fuzzy semistar operations [6] and fuzzy semistar operations of finite character [5] have been introduced to provide a natural and general setting of common treatment and characterizations of integral domains in the context of fuzzy theory. For an overview of fuzzy star operations and fuzzy semistar operations on integral domains, the reader may refer to [5–7].

This paper is the sequel of our papers [5,6] aiming to develop more tools that help advance the study of fuzzy star operations in fuzzy multiplicative ideal theory. For instance in [5], we described the set of all fuzzy semistar of finite character on valuation domains. Note that valuation domains are particular cases of Prüfer domains. In this present work, we give a complete description of the set of all fuzzy semistar operations on Prüfer domains. We show how a

E-mail address: oheubokw@svsu.edu.

http://dx.doi.org/10.1016/j.fss.2017.03.002
0165-0114/© 2017 Elsevier B.V. All rights reserved.

fuzzy semistar operation on an integral domain R induces canonically a fuzzy semistar operation on an overring T of R and vice versa. As an application, we obtain new characterizations of Prüfer domains and conducive domains.

For easy reference, in Section 2, we review some definitions and preliminary results from [5–8] to be used in later sections.

In Section 3, we show how a fuzzy semistar operation on an integral domain R induces canonically a fuzzy semistar operation on an overring T of R and how a fuzzy semistar operation on T induces canonically a semistar operation on R. We also study some interplays between these fuzzy semistar operations. Precisely, we show that finite character, cancellation, stability, a.b., and e.a.b. are all preserved when one passes from a fuzzy semistar operation on an integral domain R to an induced canonical fuzzy semistar operation on an overring T of R. We also show that the fuzzy semistar on R canonically induced from a fuzzy semistar operation on T preserves finite character, cancellation, a.b., e.a.b., but not stability. However, we show that stability is preserved under the assumption that T is a flat overring of R.

In Section 4, we give some characterizations of Prüfer domains R in terms of their overrings T and the induced bijection between the set of all stable fuzzy semistar operations $\star$ on R such that $\chi_T = \chi_R^\star$ and the set of all stable fuzzy (semi)star operations on T. Also, a complete description of the set of all fuzzy semistar operations of finite character on Prüfer domains is obtained. Finally, we provide some characterizations of conducive domains in terms of valuation overrings and fuzzy semistar operations.

2. Preliminaries and notations

Zadeh in [13] introduced the notion of a fuzzy subset of a non-empty set X as a function from X to $[0, 1]$. Goguen in [3] generalized the notion of fuzzy subset of X to that of an L-fuzzy subset, namely a function from X to a lattice L. Throughout this paper, we let R be an integral domain with quotient field K, and L stands for a complete lattice that satisfies the infinite distributive law with least element 0 and greatest element 1.

Recall that an *integral domain* R is a commutative ring with identity and no zero-divisors. Hence its quotient ring K is a field. An *ideal* of an integral domain R is a nonempty subset I of R such that $x - y \in I$ and $rx \in I$ for all $x, y \in I$ and $r \in R$. A group $(M, +)$ is an R-*module* if there is a mapping $R \times M \to M$, $(r, x) \mapsto rx$, satisfying the following conditions: $1x = x$; $r(x - y) = rx - ry$; and $(rt)x = r(tx)$ for all $r, t \in R$ and $x, y \in M$, where 1 is the identity of R. Note that the quotient field K of an integral domain R is an R-module. An R-*submodule* N of an R-module M is a subgroup of M such that $rx \in N$ for all $r \in R$ and $x \in N$. An R-submodule of K is just then a submodule of the R-module K. For instance, any ring T such that $R \subseteq T \subseteq K$ is an R-submodule of K. We denote $\overline{F}(R)$ the set of all nonzero R-submodules of K, $F(R)$ the set of all nonzero fractional ideals of R, i.e., all $A \in \overline{F}(R)$ such that $dA \subseteq R$ for some nonzero $d \in R$, and $f(R)$ the set of all nonzero finitely generated R-submodules of K. For an overview of integral domains and modules the reader may refer to [2,11].

An L-*fuzzy subset* of R is a function from R into L. For simplicity, as L is fixed, we use fuzzy for L-fuzzy. Let μ, ν be fuzzy subsets of R. We write $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)$ for all $x \in R$. If $\mu \subseteq \nu$ and there is an $x \in R$ such that $\mu(x) < \nu(x)$, we write $\mu \subset \nu$. We denote the image of μ by $Im(\mu)$. We say that μ is *finite-valued* if $|Im(\mu)| < \infty$ (in this paper, $|X|$ denotes the cardinality of X). Let $\mu_t = \{x \in R | \mu(x) \geq t\}$, μ_t is called a *level subset* of μ. We let χ_A denote the characteristic function of the subset A of R and let $\chi_R^{(t)}$ denote the fuzzy subset of K such that $\chi_R^{(t)}(x) = 1$ if $x \in R$ and $\chi_R^{(t)}(x) = t$ if $x \in K \setminus R$, for each $t \in L \setminus \{1\}$.

A fuzzy subset μ of R is a *fuzzy ideal* of R if for every $x, y \in R$, $\mu(x - y) \geq \mu(x) \wedge \nu(y)$ and $\mu(xy) \geq \mu(x) \vee \nu(y)$. A fuzzy subset μ of R is a fuzzy ideal of R if and only if $\mu(0) \geq \mu(x)$ for every $x \in R$ and μ_t is an ideal of R for every $t \in L$. A fuzzy subset β of K is a *fuzzy* R-*submodule* of K if $\beta(x - y) \geq \beta(x) \wedge \beta(y)$ and $\beta(rx) \geq \beta(x)$ and $\beta(0) = 1$, for every $x, y \in K$, $r \in R$. A fuzzy subset β of K is a fuzzy R-submodule of K if and only if $\beta(0) = 1$ and β_t is an R-submodule of K for every $t \in L$. We let β_* denote the following set $\{x \in K | \beta(x) = \beta(0)\}$. Let $\mathbb{Z}_+$ denote the set of positive integers.

Let α and β be fuzzy subsets of K. Define the fuzzy subset $\alpha \circ \beta$ of K as follows: for every $x \in K$, $(\alpha \circ \beta)(x) = \bigvee \{\alpha(y) \wedge \beta(z) | y, z \in K, x = yz\}$. The *product* of α and β, written $\alpha\beta$, is defined as follows: for each $x \in K$,

$$(\alpha\beta)(x) = \bigvee \left\{ \bigwedge_{i=1}^n (\alpha(x_i) \wedge \beta(x_i')) | x_i, x_i' \in K, n \in \mathbb{Z}_+, x = \sum_{i=1}^n x_i x_i' \right\}.$$

Let $\{\alpha_i | i = 1, \ldots, n\}$ be a finite collection of fuzzy subsets of K. The *sum* $\sum_{i=1}^n \alpha_i$ of the α_i's is the fuzzy subset of K defined as follows: for each $x \in K$,