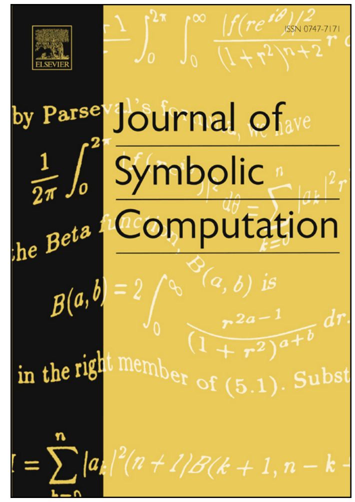
Accepted Manuscript



Formalization of the Arithmetization of Euclidean Plane Geometry and Applications

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PII: S0747-7171(18)30036-1
DOI: https://doi.org/10.1016/j.jsc.2018.04.007
Reference: YJSCO 1848

To appear in: *Journal of Symbolic Computation*

Received date: 2 August 2016
Accepted date: 4 February 2017

Please cite this article in press as: Boutry, P., et al. Formalization of the Arithmetization of Euclidean Plane Geometry and Applications. *J. Symb. Comput.* (2018), https://doi.org/10.1016/j.jsc.2018.04.007

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Abstract

This paper describes the formalization of the arithmetization of Euclidean plane geometry in the Coq proof assistant. As a basis for this work, Tarski's system of geometry was chosen for its well-known metamathematical properties. This work completes our formalization of the two-dimensional results contained in part one of the book by Schwabhäuser Szmielew and Tarski *Metamathematische Methoden in der Geometrie*. We defined the arithmetic operations geometrically and proved that they verify the properties of an ordered field. Then, we introduced Cartesian coordinates, and provided characterizations of the main geometric predicates. In order to prove the characterization of the segment congruence relation, we provided a synthetic formal proof of two crucial theorems in geometry, namely the intercept and Pythagoras' theorems. To obtain the characterizations of the geometric predicates, we adopted an original approach based on bootstrapping: we used an algebraic prover to obtain new characterizations of the predicates based on already proven ones. The arithmetization of geometry paves the way for the use of algebraic automated deduction methods in synthetic geometry. Indeed, without a "back-translation" from algebra to geometry, algebraic methods only prove theorems about polynomials and not geometric statements. However, thanks to the arithmetization of geometry, the proven statements correspond to theorems of *any* model of Tarski's Euclidean geometry axioms. To illustrate the concrete use of this formalization, we derived from Tarski's system of geometry a formal proof of the nine-point circle theorem using the Gröbner basis method. Moreover, we solve a challenge proposed by Beeson: we prove that, given two points, an equilateral triangle based on these two points can be constructed in Euclidean Hilbert planes. Finally, we derive the axioms for another automated deduction method: the area method.

Key words: Formalization, geometry, Coq, arithmetization, intercept theorem, Pythagoras' theorem, area method

⋆ Extended version of a paper presented at SCSS 2016. Section 3 contains new material.

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