

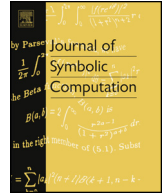


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# On the complexity of integer matrix multiplication

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## ABSTRACT

Let  $M(n)$  denote the bit complexity of multiplying  $n$ -bit integers, let  $\omega \in (2, 3]$  be an exponent for matrix multiplication, and let  $\lg^* n$  be the iterated logarithm. Assuming that  $\log d = O(n)$  and that  $M(n)/(n \log n)$  is increasing, we prove that  $d \times d$  matrices with  $n$ -bit integer entries may be multiplied in

$$O(d^2 M(n) + d^\omega n 2^{O(\lg^* n - \lg^* d)} M(\lg d) / \lg d)$$

bit operations. In particular, if  $n$  is large compared to  $d$ , say  $d = O(\log n)$ , then the complexity is only  $O(d^2 M(n))$ .

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## 1. Introduction

In this paper we study the complexity of multiplying  $d \times d$  matrices whose entries are integers with at most  $n$  bits. We are particularly interested in the case that  $n$  is very large compared to  $d$ , say  $d = O(\log n)$ . All complexity bounds refer to deterministic bit complexity, in the sense of the multi-tape Turing model (Papadimitriou, 1994).

Matrices with large integer coefficients appear naturally in several areas. One first application is to the efficient high precision evaluation of so-called holonomic functions (such as  $\exp$ ,  $\log$ ,  $\sin$ ,

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Bessel functions, and hypergeometric functions) using a divide and conquer technique (Chudnovsky and Chudnovsky, 1990; Haible and Papanikolaou, 1997; van der Hoeven, 1999, 2001, 2007). Another application concerns recent algorithms for computing the  $L$ -series of algebraic varieties (Harvey, 2014, 2015; Harvey and Sutherland, 2014, 2016; Harvey et al., 2016a). The practical running time in these applications is dominated by the multiplication of matrices with large integer entries, and it is vital to have a highly efficient implementation of this fundamental operation. Typical parameters for these applications are  $n$  around  $10^8$  bits, and  $d$  around 10.

In this paper, we focus mainly on theoretical bounds. We write  $M_d(n)$  for the cost of multiplying  $d \times d$  matrices with  $n$ -bit integer entries, and  $M(n) := M_1(n)$  for the cost of multiplying  $n$ -bit integers. We will also write  $M_{R,d}(n)$  for the algebraic complexity of multiplying  $d \times d$  matrices whose entries are polynomials of degree  $< n$  over an abstract effective ring  $R$ , and  $M_R(n) := M_{R,1}(n)$ .

Schönhage and Strassen (1971) used fast Fourier transforms (FFTs) to prove that  $M(n) = O(n \log n \log \log n)$  for large  $n$ . Fürer (2009) improved this to  $M(n) = O(n \log n 2^{O(\lg^* n)})$  where  $\lg^*$  is the iterated logarithm, i.e.,

$$\begin{aligned} \lg n &:= \lceil \log_2 n \rceil, \\ \lg^* n &:= \min\{k \in \mathbb{N} : \lg^{\circ k} n \leq 1\}, \\ \lg^{\circ k} &:= \underbrace{\lg \circ \dots \circ \lg}_{k \times} \end{aligned}$$

and this was recently sharpened to  $M(n) = O(n \log n 8^{\lg^* n})$  (Harvey et al., 2016b). The best currently known bound (Cantor and Kaltofen, 1991) for  $M_R(n)$  is  $M_R(n) = O(n \log n \log \log n)$ ; if  $R$  is a ring of finite characteristic this may be improved to  $M_R(n) = O(n \log n 8^{\lg^* n})$  (Harvey et al., 2017).

The algebraic complexity of  $d \times d$  matrix multiplication is usually assumed to be of the form  $O(d^\omega)$ , where  $\omega$  is a so-called exponent of matrix multiplication (von zur Gathen and Gerhard, 2003, Ch. 12). Classical matrix multiplication yields  $\omega = 3$ , and Strassen's algorithm (Strassen, 1969) achieves  $\omega = \log 7 / \log 2 \approx 2.807$ . The best currently known exponent  $\omega < 2.3728639$  was found by Le Gall (Le Gall, 2014; Coppersmith and Winograd, 1987).

When working over the integers and taking into account the growth of coefficients, the general bound for matrix multiplication specialises to

$$M_d(n) = O(d^\omega M(n + \lg d)).$$

Throughout this paper we will enforce the very mild restriction that  $\log d = O(n)$ . Under this assumption the above bound simplifies to

$$M_d(n) = O(d^\omega M(n)).$$

The main result of this paper is the following improvement.

**Theorem 1.** *Assume that  $M(n)/(n \log n)$  is increasing. Let  $C > 1$  be a constant. Then*

$$M_d(n) = O(d^2 M(n) + d^\omega n 2^{O(\lg^* n - \lg^* d)} M(\lg d) / \lg d), \quad (1)$$

*uniformly for  $n \geq 2$  and  $d \geq 1$ , under the condition that  $\lg d \leq Cn$ .*

In particular, if  $n$  is large compared to  $d$ , say  $d = O(\log n)$ , then (1) simplifies to

$$M_d(n) = O(d^2 M(n)). \quad (2)$$

This bound is essentially optimal (up to constant factors), in the sense that we cannot expect to do better for  $d = 1$ , and the bound grows proportionally to the input and output size as a function of  $d$ .

The new algorithm has its roots in studies of analogous problems in the algebraic complexity setting. When working over an arbitrary effective ring  $R$ , a classical technique for multiplying polynomial matrices is to use an evaluation-interpolation scheme. There are many different evaluation-interpolation strategies (van der Hoeven, 2010, Sections 2.1–2.3) such as Karatsuba, Toom–Cook, FFT,

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