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Motions of grid-like reflection frameworks

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ABSTRACT

Combinatorial characterisations are obtained of symmetric and anti-symmetric infinitesimal rigidity for two-dimensional frameworks with reflectional symmetry in the case of norms where the unit ball is a quadrilateral and where the reflection acts freely on the vertex set. At the framework level, these characterisations are given in terms of induced monochrome subgraph decompositions, and at the graph level they are given in terms of sparsity counts and recursive construction sequences for the corresponding signed quotient graphs.

1. Introduction

The objects considered in this article are geometric constraint systems where the constraints are determined by a possibly non-Euclidean choice of norm. The main results are new contributions in both geometric and combinatorial rigidity. At the geometric level, characterisations are provided for rigid two-dimensional symmetric frameworks constrained by norms with a quadrilateral unit ball (the ℓ^1 and ℓ^∞ norms for example). At the combinatorial level, the problem of deciding whether a graph can be realized as a forced symmetric or anti-symmetric isostatic reflection framework is considered and complete characterisations are obtained. Overall this article builds on recent work analyzing the rigidity of frameworks in normed linear spaces, with and without symmetry (see for example Kitson, 2015; Kitson and Power, 2014; Kitson and Schulze, 2015, 2016).

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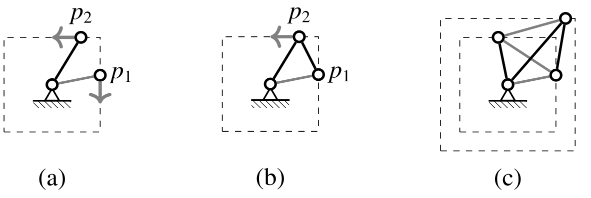

Fig. 1. Grid-like frameworks in $(\mathbb{R}^2, \|\cdot\|_\infty)$, where one of the vertices is fixed at the origin: the framework in (a) has two degrees of freedom, as p_1 and p_2 can move vertically and horizontally, respectively, independent of each other; the framework in (b) has one degree of freedom, as p_2 can still move horizontally; the framework in (c) is rigid. The colours of the edges are induced by their orientation relative to the unit ball in $(\mathbb{R}^2, \|\cdot\|_\infty)$.

A bar-joint framework in the plane is referred to as *grid-like* if the bar-lengths are determined by a norm with a quadrilateral unit ball. The allowable motions of such a framework constrain vertices adjacent to any pinned vertex to move along the boundary of a quadrilateral which is centred at the pinned vertex and obtained from the unit ball by translation and dilation (see Fig. 1). This is an important context from the point of view of applications. For example, the problem of maintaining rigid formations of mobile autonomous agents is a well-known application of geometric rigidity theory and its associated "pebble game" algorithms (see Eren et al., 2004). However, the Euclidean metric may not always be the most natural choice for controlling a formation. For instance, it may not be possible to detect Euclidean distances between agents (e.g. due to obstacles in the terrain). Moreover, if the agents have restricted mobility (e.g. with only vertical and horizontal directions of motion possible) then standard methods from Euclidean rigidity theory will have limited use. In these cases it may be desirable to have a rigidity theory for a non-Euclidean norm (such as the ℓ^1 or ℓ^∞ norm) as an alternative approach to formation control. An accompanying theory for *symmetric* frameworks may provide more efficient architectures for the control of formations due to the smaller size of the quotient graphs and their associated constraint systems.

There are three main aims of this article. The first is to formally introduce and develop symmetric and anti-symmetric infinitesimal rigidity for $\mathbb{Z}_2$-symmetric frameworks in general normed linear spaces. This is achieved in Section 2. Each infinitesimal flex is shown to decompose in a unique way as a sum of a symmetric and an anti-symmetric flex. Moreover, the rigidity operator is shown to admit a block decomposition which leads in a natural way to a consideration of orbit matrices. Sparsity counts, expressed in terms of an associated signed quotient graph, are then derived for symmetrically and anti-symmetrically isostatic frameworks. When applied to Euclidean frameworks, the block decomposition reduces to that studied in Kangwai and Guest (2000), Owen and Power (2010), Schulze (2010), while the orbit matrices and sparsity counts coincide with those in Jordán et al. (2016), Schulze and Tanigawa (2015), Schulze and Whiteley (2011).

The second aim is to characterise symmetric, anti-symmetric and general infinitesimal rigidity for grid-like frameworks with reflectional symmetry, where the reflection acts *freely* on the vertex set. In Section 3.1, characterisations are obtained in terms of edge colourings for the signed quotient graph. These edge colourings are induced from a symmetric edge-colouring of the covering graph which is in turn induced by the positioning of the framework relative to the unit ball. This may be viewed as an extension to symmetric frameworks of methods used in Kitson (2015), Kitson and Power (2014).

The third aim, which is in the spirit of Laman's theorem (see Laman, 1970; Tay, 1993; Whiteley, 1996), is to provide combinatorial characterisations for graphs which admit placements as rigid grid-like frameworks with reflectional symmetry. This is achieved in Section 3.2 for both symmetric and anti-symmetric infinitesimal rigidity. The characterisations provide the sufficiency direction for the necessary sparsity counts derived in the general theory of Section 2. The proof applies an inductive construction for signed quotient graphs together with the results of Section 3.1. Note that these matroidal counts can be checked in polynomial time using a straightforward adaptation of the algorithm described in Jordán et al. (2016, Sect. 10) (see also Berardi et al., 2011).

The results of Section 3.2 are analogous to the corresponding results for Euclidean reflection frameworks in Jordán et al. (2016), Malestein and Theran (2015). It is important to note, however, that unlike the Euclidean situation (see Schulze and Tanigawa, 2015), the respective characterisations of