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On the effective and automatic enumeration of polynomial permutation classes

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ABSTRACT

We describe an algorithm, implemented in Python, which can enumerate any permutation class with polynomial enumeration from a structural description of the class. In particular, this allows us to find formulas for the number of permutations of length n which can be obtained by a finite number of block sorting operations (e.g., reversals, block transpositions, cut-and-paste moves).

1. Introduction

The Fibonacci Dichotomy of Kaiser and Klazar (2003) was one of the first general results on the enumeration of permutation classes. It states that if there are fewer permutations of length n in a class than the nth Fibonacci number, for any n, then the enumeration of the class is given by a polynomial for sufficiently large n. Since the Fibonacci Dichotomy was established for permutation classes, Balogh et al. (2006) showed that it extends to the (more general) context of ordered graphs, while other proofs of the Fibonacci Dichotomy for permutations have been given by Huczynska and Vatter (2006) and Albert et al. (2007).

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While much of the focus on this strand of research has shifted to the consideration of larger classes (see Bollobás (2007), Klazar (2010), and Vatter (2015) for surveys), we return to consider two open questions about polynomial classes.

- **Question 1.1.** Given a structural description of a polynomial permutation class, how can we enumerate it?
- **Question 1.2.** Which polynomials occur as enumerations of polynomial classes?

We view a satisfactory answer to Question 1.1 as a prerequisite for the investigation of Question 1.2, and thus our focus in this paper is on enumerating polynomial classes from a structural description. Our answer to Question 1.1 also has applications to the study of genome rearrangements, as discussed in Section 3. In particular, the algorithm can be applied to the problem of evolutionary distance, which investigates the number of genomes of fixed mutation distance from the identity.

The permutation π of length n contains the permutation σ of length k (written $\sigma \leqslant \pi$) if π has a subsequence of length k which is order isomorphic to σ. For example, $\pi = 391867452$ (written in list, or one-line notation) contains $\sigma = 51342$, as can be seen by considering the subsequence 91672 $(= \pi(2)\pi(3)\pi(5)\pi(6)\pi(9))$. A *permutation class*, or simply *class*, is a downset in this subpermutation order; thus if $\mathcal{C}$ is a class, $\pi \in \mathcal{C}$, and $\sigma \leqslant \pi$, then $\sigma \in \mathcal{C}$.

While there are many ways to specify a class, two are particularly relevant to this problem. One is by the *basis* of the class, the minimal permutations *not* in the class. One may also specify a polynomial class by providing some structural description. We adopt this structural approach to the specification of classes.

We must first formalize the notion of the structure of polynomial classes. Following Albert and Atkinson (2005), an *interval* in a permutation is a sequence of contiguous entries whose values form an interval of natural numbers. A *monotone interval* is an interval in which the entries are monotone (increasing or decreasing). Given a permutation σ of length m and nonempty permutations $\alpha_1, \dots, \alpha_m$, the *inflation* of σ by $\alpha_1, \dots, \alpha_m$ is the permutation $\pi = \sigma[\alpha_1, \dots, \alpha_m]$ obtained by replacing each entry $\sigma(i)$ by an interval that is order isomorphic to α_i, while maintaining the relative order of the intervals themselves. For example,

$$3142[1, 321, 1, 12] = 6\ 321\ 7\ 45.$$

Going against traditional conventions, in this work we *allow inflations by the empty permutation* unless specifically forbidden.

The polynomial classes are very special cases of geometric grid classes (Albert et al., 2013), and they can therefore be described, roughly, as classes for which the entries of every member of the class can be partitioned into a finite number of monotone intervals, which are related to each other in one of a finite number of ways. To describe this more concretely, let us say that a *peg permutation* is a permutation where each entry is decorated with a $+$, $-$, or $\bullet$, such as

$$\tilde{\rho} = 3^\bullet 1^- 4^\bullet 2^+$$

As demonstrated above, we decorate peg permutations with tildes; in this context, ρ denotes for us the underlying (non-pegged) permutation, 3142 in this example.

The *grid class* of the peg permutation $\tilde{\rho}$, denoted $\mathrm{Grid}(\tilde{\rho})$, is the set of all permutations which may be obtained by inflating ρ by monotone intervals of type determined by the signs of $\tilde{\rho}$: $\rho(i)$ may be inflated by an increasing (resp., decreasing) interval if $\tilde{\rho}(i)$ is decorated with a $+$ (resp., $-$) while it may only be inflated by a single entry (or the empty permutation) if $\tilde{\rho}(i)$ is dotted. Thus $\pi \in \mathrm{Grid}(\tilde{\rho})$ if its entries can be partitioned into monotone intervals which are compatible with $\tilde{\rho}$; we refer to this as a *$\tilde{\rho}$-partition* of π.

Given a set $\tilde{G}$ of peg permutations, we denote the union of their corresponding grid classes by

$$\mathrm{Grid}(\tilde{G}) = \bigcup_{\tilde{\rho} \in \tilde{G}} \mathrm{Grid}(\tilde{\rho}).$$

As the next result shows, our goal is to enumerate such classes.