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Reversibility properties of the fire-fighting problem in graphs

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ABSTRACT

Assume that a fire spreads in an $n \times n$ grid where some fire fighters are deployed. Initially, all cells of the grid are on fire, except those occupied by a fighter. In each step, fighters can move to adjacent cells and put their fire out, while each burning cell ignites all neighbors not protected by a fighter. The question, also known as a lion-and-man problem [3], is how many fighters are needed to completely extinguish the fire. A column of n fighters can sweep the grid and erase the fire. Whether $n-1$ fighters are sufficient is still an open problem.

This note presents the following structural property that holds for fire-fighting in arbitrary undirected graphs, including grid graphs as a special case.

Suppose the fighters perform a sequence of moves, M, that transform configuration A into configuration B. If we swap the states of all vertices in B that do not contain a fighter and let the fighters run backwards, we reach a configuration where each vertex burning in A is now free of fire. As a consequence, if M is an extinguishing strategy, so is its reverse, $\overline{M}$.

1. Introduction

Let us consider an $n \times n$ grid and k fire-fighters placed on some of the cells. Each cell has (at most) four neighbors. Initially, all cells not occupied by a fighter are on fire. In any time step,

- each fire-fighter can move to a neighboring cell and extinguish the fire possibly burning there;
- simultaneously, the fire spreads from each burning cell to each neighboring cell that is not occupied by a staying or newly arriving fire-fighter;
- as a fighter moves from cell c to an adjacent burning cell c', the fire cannot spread from c' to c at the same time (but it spreads from c' to all other neighbors of c subject to the previous rule).

An example is shown in Fig. 1. Disks mark cells on fire, crosses indicate fighters, and “clean” cells without fire are blank.

The fire-fighters' task is to completely extinguish the fire. The question is how many fighters are needed to this end. Initially, the problem was posed as a lion-and-man problem, where a man tries to escape from a number of lions. The lions do not know the man's initial position nor his actual movements. They would like to catch the man, and therefore try to reduce to zero the set of positions remaining open to him.

Obviously, in an $n \times n$ grid, n fighters are sufficient; they can form a column and simply sweep the grid. Dumitrescu et al. [3] established a lower bound of $\sqrt{n}$. Later, a lower bound of $\frac{n}{2}$ was obtained by Brass et al. [2] and Berger et al. [1], using isoperimetric inequalities. The gap $(n/2, n)$ is still open, despite serious efforts.

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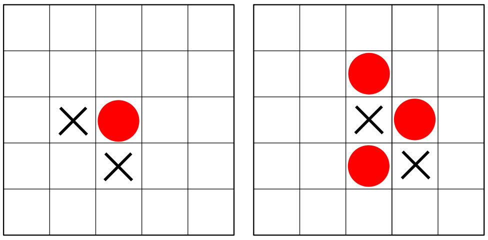

Fig. 1. Both fighters are moving one step to the right.

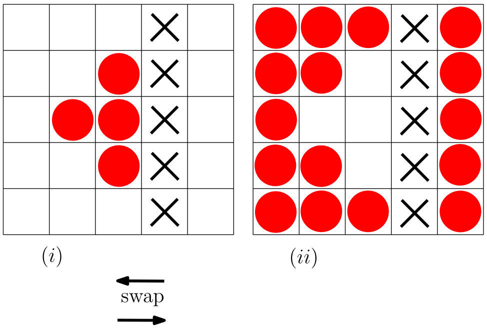

Fig. 2. In configuration (i), if fighters move one step to the left, the status of all cells are swapped, and fighters move back to the right, configuration (ii) results.

The purpose of this short note is to present a structural property that may help to cast more light on the problem. This property holds for fire-fighting in arbitrary undirected graphs, including grids.

2. Moves and swaps

We take our motivation from Fig. 2. Configurations (i) and (ii) result from each other by swapping the status of all cells not occupied by a fighter. As the fighter column in (i) moves one step to the left, the formation of four burning cells moves to the left, too. Similarly, in (ii) the four clean cells move to the right, as the fighter column does. Thus, for configuration (i) we obtain the identity

$$\text{right} \circ \text{swap} \circ \text{left} = \text{swap}. \tag{1}$$

Here, the sequence on the left hand side is left followed by swap followed by right.

Formula 1 does not hold when applied to configuration (ii). In fact, after executing the operations on the left hand side the fire is extinguished.

For general undirected graphs, we have the following result, stated for vertices rather than cells.

Theorem 1. *Suppose the fighters perform a sequence of moves that transform configuration A into B. Now the status of all vertices in B not occupied by a fighter are swapped, and the fighters perform all moves backwards. In the resulting configuration, all vertices that were burning in A are now free of fire.*

The proof of Theorem 1 is based on the following lemma.

Lemma 1. *Let vertex c of configuration A be on fire. Suppose the fighters make a single move m, the status of all vertices is swapped, and the fighters perform the reverse move $\overline{m}$. Then, vertex c is free of fire.*

Proof. We refer to Fig. 3 where the statements of the proof are listed near the configurations A to D in which they hold, numbered by order of appearance. Initially, vertex c burns, by assumption, (1). After the fighters' move m it still burns unless a fighter has moved to c, (2). In the first case, c will be clean after the swap, (3). Now we consider the status of c after the reverse move, $\overline{m}$. In D, vertex c cannot contain a fighter because it was burning in A. If it is clean, we are done, (4). Otherwise, it must be burning, (5). Since c was not burning in configuration C, by (3), a burning neighbor d has ignited c during move $\overline{m}$, (6a); for this to happen, no fighter can have moved from c to d during $\overline{m}$, (6b). Therefore, in configuration B, vertex d is a clean neighbor of c, (7a), and no fighter has moved in m from d to c, (7b). By fact (7a), vertex d is clean in B; so d must have been clean in A, or it has been left, during m, by a fighter that has not moved into c, by (7b). But then vertex c, which was burning in A, has ignited d during m—a contradiction to (7a). □