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On sufficient conditions for mean residual life and related orders

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ABSTRACT

A well known sufficient condition for the mean residual life order of two random variables is the hazard rate order of the two random variables. The hazard rate order is characterized by the monotonicity of the ratio of the two survival functions. However in many cases this ratio is non monotone and the hazard rate order does not hold. The purpose is to show that, in some situations, this non monotonicity is still a sufficient condition for the mean residual life order, under some additional mild conditions. Applications to compare some parametric models of distributions and generalized order statistics are provided. Similar results are given for the mean inactivity time order.

1. Introduction

The comparison of random quantities in terms of the so called "stochastic orders" has received a great attention along the last 40–50 years and we can find applications of this topic in different fields, like reliability, insurance, risks, finance, epidemics and so on (see Müller and Stoyan (2002), Denuit et al. (2005) and Shaked and Shanthikumar (2007)). This topic deals with several criteria to compare two random quantities, in order to select which one is "larger", according to dispersion, residual lifetimes, concentration and so on. Among the most important orders we have the hazard rate and mean residual life orders. It is well known that a sufficient condition for the mean residual life order of two random variables is the hazard rate order of the two random variables. The hazard rate order is characterized by the monotonicity of the ratio of the survival functions. However in many cases this ratio is non monotone and the hazard rate order does not hold. The purpose of this paper is to show that, in some situations, this non monotonicity is still a sufficient condition for the mean residual life order, under some additional mild conditions. In Section 2 some examples are considered to provide a motivation for the main result of this section about sufficient conditions for the mean residual life order. We also explore further the relationship among these conditions and the $\overline{P}-\overline{P}$ plot. Additionally we show the relationship of this condition and the non monotonicity of the ratio of the densities. We provide also some parametric examples of distributions where the results can be applied. Next, in Section 3, we apply the new results to the comparison of generalized order statistics. In Section 4 we provide a more general result. To finish, in Section 5, we develop a similar study, without proofs, for the mean inactivity time order.

In this paper "increasing" and "decreasing" mean "nondecreasing" and "nonincreasing", respectively. For any random variable X and any event A, we use $\{X|A\}$ to denote the random variable whose distribution is the conditional distribution of X given A. Also given a random variable X with continuous distribution function F and density f, $\bar{F} \equiv 1 - F$ denotes the survival function, $f/\overline{F}$ is the hazard rate, f/F is the reversed hazard rate and u_X denotes the right extreme of the support. By $=_{st}$ we denote equality in law.

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2. A sufficient condition for the mean residual life order

Let us recall first the definitions of some stochastic orders of interest along the paper. First we consider two stochastic orders related to the comparison in some probabilistic sense of residual lifetimes (see Shaked and Shanthikumar (2007) for definitions and results provided in this section).

Definition 2.1. Let X and Y be two random variables, with continuous distribution functions F and G respectively. We say that:

(i) X is less than Y in the hazard rate order, denoted by $X \leq_{\text{hr}} Y$, if

$$\frac{\overline{G}(x)}{\overline{F}(x)} \quad \text{is increasing in } x \in (-\infty, \max(u_X, u_Y)),$$

where $a/0$ is taken to be equal to ∞ whenever $a > 0$ or equivalently if, and only if,

$$\overline{F}(x)\overline{G}(y) \geq \overline{F}(y)\overline{G}(x) \quad \text{for all } x \leq y.$$

(ii) X is less than Y in the mean residual life order, denoted by $X \leq_{\text{mrl}} Y$, if

$$\frac{\int_x^{+\infty} \overline{G}(u)du}{\int_x^{+\infty} \overline{F}(u)du} \quad \text{is increasing in } x \in \left\{ x \,\middle|\, \int_x^{+\infty} \overline{F}(u)du > 0 \right\},$$

or equivalently, if, and only if,

$$\overline{F}(x) \int_x^{+\infty} \overline{G}(u)du \geq \overline{G}(x) \int_x^{+\infty} \overline{F}(u)du \quad \text{for all } x.$$

The definition of the mean residual life order in the discrete case should be modified and therefore in this paper we only deal with continuous random variables (see Shaked and Shanthikumar (2007, pp. 82–83)).

The reason why these two criteria are based on the comparison of residual lifetimes is the following. Let us consider first the notion of residual lifetimes and the usual stochastic order. Given a random variable X and given a value x such that $\overline{F}(x) > 0$ then the residual life of X given x is the conditional random variable $\{X - x | X > x\}$. Now we consider the definition of the stochastic and increasing convex orders.

Definition 2.2. Let X and Y be two random variables, with distribution functions F and G respectively. We say that X is less than Y in the stochastic order, denoted by $X \leq_{\text{st}} Y$, if

$$\overline{F}(x) \leq \overline{G}(x) \quad \text{for all } x \in \mathbb{R}.$$

The increasing convex order can be defined in terms of the stop-loss function. Given a random variable X the stop-loss function for any point $x \in \mathbb{R}$, is defined as $E[(X - x)_+] = \int_x^{+\infty} \overline{F}(u)du$ where $(x)_+ = x$ if $x \geq 0$ and $(x)_+ = 0$ if $x < 0$, and it will be denoted by $\pi_X(x)$, that is, $\pi_X(x) \equiv E[(X - x)_+]$. The stop-loss function is well known in the context of actuarial risks.

Definition 2.3. Let X and Y be two random variables. We say that X is less than Y in the increasing convex order, denoted by $X \leq_{\text{icx}} Y$, if

$$\pi_X(x) \leq \pi_Y(x) \quad \text{for all } x \in \mathbb{R}.$$

It is well known that the hazard rate order is stronger than the stochastic order, and the stochastic order is stronger than the increasing convex order, that is,

$$X \leq_{\text{hr}} Y \Rightarrow X \leq_{\text{st}} Y \Rightarrow X \leq_{\text{icx}} Y. \tag{2.1}$$

The following characterizations of the hazard rate and mean residual life orders are well known and highlight the role of residual lifetimes for these two orders. Given two random variables X and Y, with continuous distribution functions F and G respectively, then

(i) $X \leq_{\text{hr}} Y$, if and only if,

$$\{X - x | X > x\} \leq_{\text{st}} \{Y - x | Y > x\} \quad \text{for all } x \text{ such that } \overline{F}(x), \overline{G}(x) > 0,$$

or equivalently,

$$\{X - x | X > x\} \leq_{\text{hr}} \{Y - x | Y > x\} \quad \text{for all } x \text{ such that } \overline{F}(x), \overline{G}(x) > 0.$$

These results are easy to prove and the first one can be found in Shaked and Shanthikumar (2007).