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Decomposing highly connected graphs into paths of length five

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ABSTRACT

Barát and Thomassen (2006) posed the following decomposition conjecture: for each tree T, there exists a natural number k_T such that, if G is a k_T-edge-connected graph and $|E(G)|$ is divisible by $|E(T)|$, then G admits a decomposition into copies of T. In a series of papers, Thomassen verified this conjecture for stars, some bistars, paths of length 3, and paths whose length is a power of 2. We verify this conjecture for paths of length 5.

1. Introduction

A *decomposition* $\mathcal{D}$ of a graph G is a set $\{H_1, \ldots, H_k\}$ of pairwise edge-disjoint subgraphs of G whose union is G. If each subgraph H_i, $1 \leq i \leq k$, is isomorphic to a given graph H, then we say that $\mathcal{D}$ is an *H-decomposition* of G.

A well-known result of Kotzig (see [7,20]) states that a connected graph G admits a decomposition into paths of length 2 if and only if G has an even number of edges. Dor and Tarsi [16] proved that the problem of deciding whether a graph has an H-decomposition is NP-complete whenever H is a connected graph with at least 3 edges. It is then natural to consider special classes of graphs H, and look for sufficient conditions for a graph G to admit an H-decomposition. One class of graphs that has been studied from this point of view is that of paths, in special when the input graph G is regular. A pioneering work on this topic dates back to 1957, and although some others have followed, a number of questions remain open [14,17,18,20]. For the special case in which H is a tree, Barát and Thomassen [3] proposed the following conjecture.

Conjecture 1.1. *For each tree T, there exists a natural number k_T such that, if G is a k_T-edge-connected graph and $|E(G)|$ is divisible by $|E(T)|$, then G admits a T-decomposition.*

Barát and Thomassen [3] proved that Conjecture 1.1 in the special case T is the claw $K_{1,3}$ is equivalent to Tutte's weak 3-flow conjecture, posed by Jaeger [19]. They also observed that this conjecture is false if, instead of a tree, we consider a graph that contains a cycle.

Since 2008 many results on this conjecture have been found by Thomassen [28–32]. He has verified that this conjecture holds for paths of length 3, stars, a family of bistars, and paths whose length is a power of 2. In this paper we prove Conjecture 1.1 for paths of length 5. We will focus on the following version of Conjecture 1.1 for bipartite graphs.

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Conjecture 1.2. *For each tree T, there exists a natural number k'_T such that, if G is a k'_T-edge-connected bipartite graph and $|E(G)|$ is divisible by $|E(T)|$, then G admits a T-decomposition.*

Recently, Barát and Gerbner, and Thomassen independently proved that Conjectures 1.1 and 1.2 are equivalent. The next theorem states this result precisely.

Theorem 1.3 (*Barát–Gerbner [2]; Thomassen [31]*)**.** *Let T be a tree with m edges, where $m > 3$. The following two statements are equivalent.*

(i) *There exists a natural number k'_T such that, if G is a k'_T-edge-connected bipartite graph and $|E(G)|$ is divisible by $|E(T)|$, then G admits a T-decomposition.*
(ii) *There exists a natural number k_T such that, if G is a k_T-edge-connected graph and $|E(G)|$ is divisible by $|E(T)|$, then G admits a T-decomposition.*

Furthermore, $k_T \leq 4k'_T + 16m^{6m+1}$ and, if in addition T has diameter at most 3, then $k_T \leq 4k'_T + 16(m+1)m$.

In this paper we verify Conjecture 1.2 (and Conjecture 1.1) in the special case T is the path of length five. More specifically, we prove that $k'_{P_5} \leq 48$.

In our proof we use a generalization of the technique used by Thomassen [28] to obtain an initial decomposition into trails of length 5. Then, inspired by the edge-switching technique used in [13], we obtain a result that allows us to "disentangle" the undesired trails of this initial decomposition and construct a pure path decomposition. The main idea uses the following fact: since in a bipartite graph, a trail T of length 5 that is not a path contains a cycle C of length 4, there are two edges of C that can be switched with other elements of the decomposition in such a way that T becomes a path. In [13] only one switching is necessary to "improve" the initial decomposition, but in this paper we need a sequence of switchings to achieve this improvement.

An extended abstract [11] of this work was presented at the conference LAGOS 2015. Further improvements were obtained since then, and these are incorporated into this work. In special, a bound for k'_{P_5} was improved from 134 to 48. Moreover, we [8,10] have been able to generalize some of the ideas presented here to prove that Conjecture 1.1 holds for paths of any given length. We consider that the ideas and techniques presented in this paper are easier to be understood, and they can be seen as a first step towards obtaining more general results not only for paths of fixed length, but also for other types of results [9,12]. As the generalization is not so straightforward, we believe that those interested on the more general case will benefit reading this work first.

The paper is organized as follows. In Section 2 we give some definitions, establish the notation and state some auxiliary results needed in the proof of our main result, presented in Section 4. In Section 3 we prove that a highly edge-connected graph admits a "canonical" decomposition into paths and trails of length 5 satisfying certain properties. In Section 4 we show how to switch edges between the elements of the above decomposition and obtain a decomposition into paths of length 5. We finish with some concluding remarks in Section 5.

2. Notation and auxiliary results

The basic terminology and notation used in this paper follows [6,15]. A *graph* has no loops or multiple edges. A *multigraph* may have multiple edges but no loops. A *directed graph* (resp. *directed multigraph*) is a graph (resp. multigraph) together with an orientation of its edges. More precisely, a directed graph (resp. multigraph) is a pair $\vec{G} = (V, A)$ consisting of a vertex-set V and a set A of ordered pairs of distinct vertices, called *directed edges* (or, simply, *edges*). When a pair (V, A) that defines a (directed) graph G is not given explicitly, such a pair is assumed to be $(V(G), A(G))$. Given a directed graph $\vec{G}$, the set of edges obtained by removing the orientation of the directed edges in $A(\vec{G})$ is denoted by $\hat{A}(\vec{G})$ and is called the *underlying edge-set* of $A(\vec{G})$. We denote by G the *underlying graph* of $\vec{G}$, that is, the graph with vertex-set $V(\vec{G})$ and edge-set $\hat{A}(\vec{G})$. We say that $\vec{G}$ is k-edge-connected if G is k-edge-connected. We denote by $G = (A \cup B, E)$ a bipartite graph G on vertex classes A and B.

We denote by $Q = v_0v_1 \cdots v_k$ a sequence of vertices of a graph G such that $v_iv_{i+1} \in E(G)$, for $i = 0, \dots, k-1$. If the edges v_iv_{i+1}, $i = 0, \dots, k-1$, are all distinct, then we say that Q is a *trail*; and if all vertices in Q are distinct, then we say that Q is a *path*. The *length* of Q is k (the number of its edges). A path of length k is denoted by P_k, and is also called a *k-path*. Note that this notation is not standard. If $\vec{Q} = v_0v_1 \cdots v_k$ is a sequence of vertices of a directed graph $\vec{G}$, we say that $\vec{Q}$ is a *path* (resp. *trail*) if Q is a path (resp. trail) in G.

We say that a directed graph $\vec{H}$ is a *copy* of a graph G if H is isomorphic to G. We say that a set $\{H_1, \dots, H_k\}$ of graphs is a *decomposition* of a graph G if $\bigcup_{i=1}^{k} E(H_i) = E(G)$ and $E(H_i) \cap E(H_j) = \emptyset$ for all $1 \leq i < j \leq k$. For a directed graph, the definition is analogous. Let $\mathcal{H}$ be a family of graphs. An *$\mathcal{H}$-decomposition* $\mathcal{D}$ of $\vec{G}$ is a decomposition of $\vec{G}$ such that each element of $\mathcal{D}$ is a copy of an element of $\mathcal{H}$. If $\mathcal{H} = \{H\}$ we say that $\mathcal{D}$ is an *H-decomposition*.

In what follows, we present some concepts and auxiliary results that will be used in the forthcoming sections. We assume here that the set of natural numbers does not contain zero.