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On chordal graph and line graph squares

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ABSTRACT

In this work we investigate the chordality of squares and line graph squares of graphs. We prove a sufficient condition for the chordality of squares of graphs not containing induced cycles of length at least five. Moreover, we characterize the chordality of graph squares by forbidden subgraphs. Transferring that result to line graphs allows us to characterize the chordality of line graph squares.

1. Introduction

A graph $G = (V, E)$ is said to be *chordal* if every cycle C of length $n \geq 4$ in G has a *chord*, i.e., there is an edge $e \in E$ connecting two nonconsecutive vertices of the cycle. As usual, a *cycle* is defined as an alternating sequence of vertices and edges $(v_1, e_1, v_2, e_2, \ldots, v_n, e_n)$ with $v_i \in V$, $e_i \in E$ for $i = 1, \ldots, n$, and $e_i \cap e_{i+1 \bmod n} = \{v_i\}$.

Due to their strong combinatorial properties, chordal graphs are one of the most extensively studied graph classes in Graph Theory and Discrete Optimization. In this work, we consider graph transformations, namely graph powers and line graphs, and their effect on chordality.

We define the *kth graph power of* G by

$$G^k := (V, \{xy \mid x, y \in V \text{ and } 1 \leq \mathrm{dist}_G(x, y) \leq k\}),$$

with $\mathrm{dist}_G(x, y)$ we denote the length of a shortest path between x and y. Now, obvious questions concerning graph powers and chordality are:

- Is the kth power G^k of a graph G chordal?
- Is the kth power G^k of a chordal graph G chordal?

Both questions were already discussed in the literature and in Section 2 we summarize several related results. After that, we will state a new sufficient condition for the chordality of a graph square and characterize the class of graphs with chordal squares. We say that two edges of a graph G are *adjacent*, if they are not disjoint. The *line graph* of a graph G is given by $L(G) := (E, \{ef \mid e \text{ and } f \text{ are adjacent}\})$ and, in Section 3, we characterize the class of graphs with a chordal line graph square.

As a consequence, algorithms designed for chordal graphs can be applied to the square and line graph square of graphs from our classes. Thus, we can solve hard optimization problems efficiently in these classes, e.g., optimum colorings with distance conditions such as strong edge colorings.

The results of this work were also presented in the master's thesis of the second author (see [8]).

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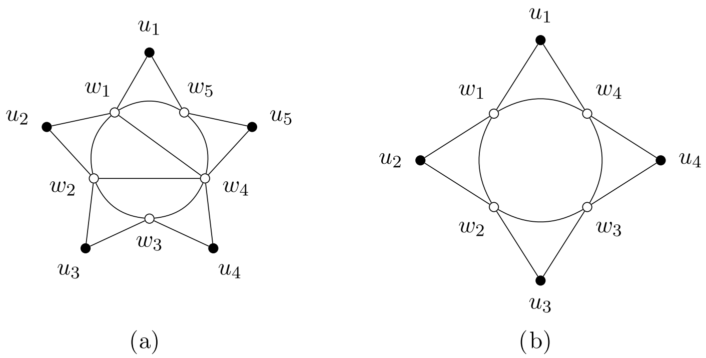

Fig. 1. (a) Example of a chordal graph whose square is not chordal. (b) The graph F_4 - a "nonchordal sunflower" of size 4.

2. Chordal squares of graphs

The *diameter* of a connected graph G is given by

$$\mathrm{dm}(G) := \max_{v,w\in V(G)} \mathrm{dist}_G(v, w),$$

while the diameter of a general graph is given by the maximum diameter of its components. Obviously, all components of $G^{\mathrm{dm}(G)}$ are complete and it is straight forward to show that $\mathrm{dm}(G)$ is in fact the smallest power for which all components of G become complete.

Lemma 2.1. *All components of G^k are complete graphs if and only if $k \geq \mathrm{dm}(G)$.*

In general, chordal graphs are not closed under graph powers (compare Fig. 1(a)). Nevertheless it is possible to show results for odd graph powers. In 1980 Laskar and Shier (see [6]) showed that G^3 and G^5 are chordal if G is chordal and they conjectured that every odd power of a chordal graph is chordal. Duchet (see [4]) proved an even stronger result which led to a number of so called Duchet-type results on graph powers.

Theorem 2.2 ([4])**.** *Let $k \in \mathbb{N}$. If G^k is chordal, so is G^{k+2}.*

Duchet's result was utilized by Laskar and Shier to characterize the class of chordal graphs which are closed under graphs powers. Namely, they characterized chordal graphs whose squares are also chordal. In order to state their result, we need some definitions. For a subset $U \subseteq V$ we define

$$G[U] := (U, \{e \in E \mid e \subset U\})$$

as the subgraph induced by U. With this notion we can rephrase the definition of chordality: G is chordal if and only if it does not contain an induced cycle, i.e., there is no subset $U \subseteq V$ such that the graph $G[U]$ is isomorphic to a cycle C of length $l \geq 4$.

Definition 2.3 (*Sunflower*)**.** A *sunflower* of size n is a graph $S = (U \cup W, E)$ with $U = \{u_1, \ldots, u_n\}$ and $W = \{w_1, \ldots, w_n\}$ such that

- $G[W]$ is chordal,
- U is a stable set, i.e., a set of pairwise nonadjacent vertices, and
- $u_i v_j \in E$ if and only if $j = i$ or $j = i + 1 \pmod n$.

The family of all sunflowers of size n is denoted by $\mathcal{S}_n$. A sunflower $S \in \mathcal{S}_n$ contained in a graph G is called *suspended* if there exists a vertex $v \notin V(S)$, such that v is adjacent to at least one pair of vertices u_i and u_j with $j \neq i \pm 1 \pmod n$. Otherwise we say S is unsuspended. (Compare Fig. 2.)

Now, we are ready to state Laskar's and Shier's result:

Theorem 2.4 ([7])**.** *Let G be chordal. Then G^2 is chordal if and only if G does not contain an unsuspended sunflower $S \in \mathcal{S}_n$ with $n \geq 4$.*

Combining the foregoing result with Duchet's Theorem leads to the family of chordal graphs that is closed under taking powers.