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Note

On linear algebra of balance-binomial graphs

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ABSTRACT

In this work, we introduce balance-binomial graphs, whose entries of adjacency matrix are balance-binomial coefficients. Then we obtain some characteristic properties of the graph.

1. Introduction

Graph theory, recently, is one of the most studied areas. Since graphs are visual objects, analysis of large graphs often requires computer support. In other words, to find out graph properties, matrix representation is used. The *adjacency matrix* A of a graph whose entries are defined as

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{if } v_i \text{ is not adjacent to } v_j, \end{cases}$$

helps us to represent a graph as a matrix.

Matrix theory that combines number theory, linear algebra, combinatorics and graph theory, is one of the most popular areas in mathematics. In the development of matrix theory, matrix operations play a very important role. The determinant, the permanent, and the inverse computation are the most used matrix operations in matrix theory. The definitions of *determinant* and the *permanent of* n-square matrix $A = [a_{ij}]$ are given as,

$$\det(A) = \sum_{\sigma \epsilon S_n} sgn(\sigma) \prod_i a_{i\sigma(i)}$$

and

$$per(A) = \sum_{\sigma \epsilon S_n} \prod_i a_{i\sigma(i)}.$$

The *Kronecker product* of two matrices $A = [a_{ij}] \in M_{m,n}(\mathsf{F})$ and $B = [b_{ij}] \in M_{p,q}(\mathsf{F})$, in a field F, is defined as

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

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The Kronecker product of matrices, also called direct product or tensor product, named after the German mathematician Leopold Kronecker, has been used since the nineteenth century and applied widely in matrix theory, statistics, signal processing and linear algebra. Andrews and Kane [2] showed the Kronecker applications to computer implementations. Many properties of the product, such as the trace, the determinant, the eigenvalues, some decompositions, have been discovered [5]. The Kronecker product has a lot of interesting properties, some well-known properties can be given as below:

Let $A \in M_n$ and $B \in M_m$:

- $\det(A \otimes B) = (\det A)^n (\det B)^m = \det(B \otimes A)$,
- $\operatorname{per}(A \otimes B) = \operatorname{per}(B \otimes A)$,
- $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ (if A and B are nonsingular).

1.1. Balancing numbers and balance-binomial graphs

The *balancing numbers*, defined by the recurrence relation

$$B_n = 6B_{n-1} - B_{n-2}, \quad \text{with initial values} \quad B_1 = 1 \text{ and } B_2 = 6,$$

have huge amount of interest. Behera and Panda [4] got some spectacular properties of balancing numbers. Moreover, Panda [8] established some characteristics of the balancing numbers with well-known number sequences. By exploiting some useful results in [8], we can give the following lemma.

Lemma 1 ([8]). *For $k = 1, 2, \ldots, m-2$,*

(i) $B_{2n} = 2B_n\sqrt{8B_n^2 + 1}$

(ii) $B_m = B_{k+1}B_{m-k} - B_kB_{m-k-1}$.

The analysis and some additional properties of balancing numbers can be seen in [7,9–11] and their references.

Let us give the following properties for balancing numbers.

Lemma 2. $B_{2^i} \equiv 3 \cdot 2^i (mod 2^{i+2})$.

Proof. By using Principle of Mathematical Induction (PMI), it is obvious the equation holds for $i = 1$. Assume that

$$B_{2^k} \equiv 3 \cdot 2^k (mod 2^{k+2}) \Longleftrightarrow B_{2^k} = 3 \cdot 2^k + m \cdot 2^{k+2}, \ m \in \mathbb{Z} \tag{1}$$

is true. So, we need to show that it is also verified for $k + 1$. By Lemma 1, we can write down

$$B_{2n} = 2B_n\sqrt{8B_n^2 + 1} \Longrightarrow B_{2n}^2 = 4B_n^2(8B_n^2 + 1).$$

By using (1), it is easy to see that the equation holds for $k + 1$. ■

Lemma 3. *For $m, n \in \mathbb{Z}$, $B_{m-n} = B_mB_{n+1} - B_nB_{m+1}$.*

Proof. For $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$

$$\begin{aligned}
B_mB_{n+1} - B_nB_{m+1} &= \frac{\left(\lambda_1^m - \lambda_2^m\right)\left(\lambda_1^{n+1} - \lambda_2^{n+1}\right) - \left(\lambda_1^n - \lambda_2^n\right)\left(\lambda_1^{m+1} - \lambda_2^{m+1}\right)}{(\lambda_1 - \lambda_2)^2} \\
&= \frac{\lambda_1^n\lambda_2^{m+1} - \lambda_1^m\lambda_2^{n+1} + \lambda_1^{m+1}\lambda_2^n - \lambda_1^{n+1}\lambda_2^m}{(\lambda_1 - \lambda_2)^2} \\
&= \frac{\lambda_1^m\lambda_2^n - \lambda_1^n\lambda_2^m}{\lambda_1 - \lambda_2} \\
&= \frac{\lambda_1^n\lambda_2^n\left(\lambda_1^{m-n} - \lambda_2^{m-n}\right)}{\lambda_1 - \lambda_2} \\
&= \frac{\lambda_1^{m-n} - \lambda_2^{m-n}}{\lambda_1 - \lambda_2} \\
&= B_{m-n}. \quad \blacksquare
\end{aligned}$$

Akbulak, Kale and Oteles [1] introduced Fibonomial graphs, denoted by G_n, whose entries are dependent on the Fibonomial coefficients. In other words, the Fibonomial graph $G_n = (V_n, E_n)$ with $3 \cdot 2^n$ vertices is a graph with vertex set $V_n = \left\{v_t : t = 0, 1, 2, \ldots, 3 \cdot 2^{n-1}\right\}$ and $E_n = \left\{\left(v_i, v_j\right) : F_{[i+j,j]} \equiv 1(mod 2)\right\}$ is the edge set. Then, the connectivity properties and the energy of the graph were investigated.

Taking into account the binomial coefficients, we define the *balance-binomial* coefficients, for $n \geq k \geq 1$, as

$$\binom{n}{k}_B = \frac{B_nB_{n-1}\cdots B_{n-k+1}}{B_1B_2\cdots B_k}, \tag{2}$$

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