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Note

On the spanning connectivity of tournaments

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ABSTRACT

Let D be a digraph. A k-container of D between u and v, $C(u, v)$, is a set of k internally disjoint paths between u and v. A k-container $C(u, v)$ of D is a strong (resp. weak) k^*-container ($k \geq 2$) if there is a set of k internally disjoint paths with the same direction (resp. with different directions allowed) between u and v and it contains all vertices of D. A digraph D is k^*-strongly (resp. k^*-weakly) connected if there exists a strong (resp. weak) k^*-container between any two distinct vertices for $k \geq 2$. Specially, we define D is 1^*-connected if D is weakly Hamiltonian connected (a 1^*-connected digraph is 1^*-strongly and also 1^*-weakly connected.) We define the strong (resp. weak) spanning connectivity of a digraph D, $\kappa_s^*(D)$ (resp. $\kappa_w^*(D)$), to be the largest integer k such that D is ω^*-strongly (resp. ω^*-weakly) connected for all $1 \leq \omega \leq k$. In this paper, we show that for $k \geq 0$, a $(2k + 1)$-strong tournament is $(k + 2)^*$-weakly connected and that for $k \geq 2$, a $2k$-strong tournament is k^*-strongly connected. Furthermore, we show that in a tournament with n vertices and irregularity $i(T) \leq k$, if $n \geq 6t+5k$ $(t \geq 2)$, then $\kappa_s^*(T) \geq t$ and if $n \geq 6t+5k-3$ $(t \geq 2)$, then $\kappa_w^*(T) \geq t + 1$.

1. Introduction

For terminology not explicitly introduced here, we refer to [2,5]. A digraph D consists of a set $V(D)$ of vertices and a set $A(D)$ of ordered pairs xy of distinct vertices called arcs, if xy is an arc of D, we say that x dominates y. If A and B are two subsets of $V(D)$ and every vertex of A dominates each vertex of B, we say that A dominates B. A directed path P is a sequence of vertices $v_1, v_2, \ldots, v_k$ in a digraph such that $v_i v_{i+1}$ is an arc for all $i = 1, 2, \ldots, k - 1$ (In this paper, all of the paths mean directed paths). The vertex v_1 is called the start of P and v_k is called the end of P. The length of P is the number of arcs on P. The vertices $v_2, v_3, \ldots, v_{k-1}$ are the internal vertices of P. Two paths are said to be internally disjoint if their internal vertices are distinct. If x and y are two vertices of D and P is a directed path from x to y, we say that P is an (x, y)-path. If $uv \in A(D)$ and P is a (u, v)-path of length $k \geq 2$, then P is called a *k-bypass* of uv.

A tournament T is a digraph such that each pair of vertices is joined by precisely one arc. A Hamiltonian path (resp. cycle) of a tournament T is a path (resp. cycle) including all vertices of T. A digraph D is *weakly Hamiltonian connected* if there exists a Hamiltonian path between any two given vertices of D. For a vertex set X of T, we define $T[X]$ as the sub-tournament induced by X. The out-neighborhood $N_T^+(x)$ of a vertex x is the set of vertices dominated by x, and the in-neighborhood $N_T^-(x)$ is the set of vertices dominating x. We denote the out-degree and in-degree of vertex x by $d^+(x) = |N_T^+(x)|$ and $d^-(x) = |N_T^-(x)|$ respectively. We say that T is k-regular if for every vertex x of T, we have $d_T^+(x) = d_T^-(x) = k$. The *irregularity* $i(T)$ of a tournament T is the maximum $|d^+(x) - d^-(x)|$ over all vertices x of T. Clearly, a tournament is regular if $i(T) = 0$. If $i(T) \neq 0$, T contains a vertex of out-degree at least $\frac{1}{2}(n-1)$ and a vertex of in-degree at least $\frac{1}{2}(n-1)$. Every vertex of T has out-degree at least $\frac{1}{2}(n - 1 - i(T))$ and in-degree at least $\frac{1}{2}(n - 1 - i(T))$.

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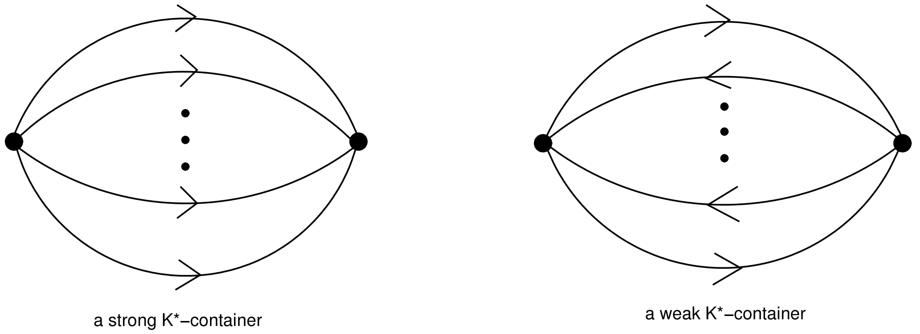

Fig. 1. Strong and weak k^*-containers.

A digraph D is strongly connected (or just strong) if there exists an (x, y)-path and a (y, x)-path for any two vertices x and y of D. We say that D is k-strongly connected (or called k-strong) if it remains strongly connected after the removal of any set of fewer than k vertices. Connectedness is a fundamental notion in graph theory, and there are countless theorems which involve it. Perhaps the most important of these is Menger's theorem [13] (vertex-connectivity), which say that a digraph D is k-strongly connected (or called k-strong) if and only if for any two vertices x and y of D, it contains k internally disjoint paths from x to y and k internally disjoint paths from y to x.

When T is strong, T has a Hamiltonian cycle [3]. When T is not strong, let $T_1, T_2, \ldots, T_k$ be the strongly connected components. Without loss of generality, we may assume that whenever $i < j$, each vertex of T_i dominates each vertex of T_j. We refer to T_i as the i'th component of T, and to T_1 and T_k as the initial and terminal components respectively.

In an undirected graph G, a k-container of G between u and v, $C(u, v)$, is a set of k internally disjoint paths between u and v. The concept of container is proposed by Hsu [4] to evaluate the performance of communication of an interconnection network. A k-container $C(u, v)$ of G is a k^*-container if it contains all vertices of G. A graph G is k^*-connected if there exists a k^*-container between any two distinct vertices of G. The study of k^*-connected graphs is motivated by the globally 3^*-connected graphs proposed by Albert et al. in [1]. Clearly, the spanning connectivity in graphs is also a generalization of Hamiltonian connectedness.

The related works about the spanning connectivity of graphs have appeared in some literatures. For examples, in [8], Lin et al. proved that the pancake graph P_n is w^*-connected for any w with $1 \leq w \leq n-1$ if and only if $n \neq 3$. In [11], Lin et al. proposed a sufficient condition for a graph to be r^*-connected. They proved that for all nonadjacent vertices u and v in a graph G, if $d_G(u)+d_G(v) \geq |V|+k$, G is r^*-connected for every $r \in \{1, 2, \ldots, k+2\}$ (k is a positive integer). The spanning connectivity of graphs and the spanning fan-connectivity of graphs are discussed in [9,12], respectively. The spanning connectivity for the line graphs and the power of a graph are proposed in [7,14], respectively. Recently, the spanning connectivity for the arrangement graphs is proposed in [15]. Some extensive properties of the spanning connectivity are discussed in [6,10,17].

We explore to introduce the spanning connectivity in digraphs. Thomassen [16] studied weak Hamiltonian connectedness and strong Hamiltonian connectedness of tournaments. In this article, we define the spanning connectivity in digraphs to generalize the weak Hamiltonian connectedness.

Let D be a digraph. A k-container of D between u and v, $C(u, v)$, is a set of k internally disjoint paths between u and v. A k-container $C(u, v)$ of D is a strong (resp. weak) k^*-container ($k \geq 2$) if there is a set of k internally disjoint paths with the same direction (resp. with different directions allowed) between u and v and it contains all vertices of D (we do not care about whether it is going from u to v, or v to u), as is shown in Fig. 1. A digraph D is k^*-strongly (resp. k^*-weakly) connected if there exists a strong (resp. weak) k^*-container between any two distinct vertices for $k \geq 2$. Specially, we define D is 1^*-connected if D is weakly Hamiltonian connected (a 1^*-connected digraph is 1^*-strongly, and also 1^*-weakly connected). We define the strong (resp. weak) spanning connectivity of a digraph D, $\kappa_s^*(D)$ (resp. $\kappa_w^*(D)$), to be the largest integer k such that D is ω^*-strongly (resp. ω^*-weakly) connected for all $1 \leq \omega \leq k$.

Based on the works of Thomassen, one naturally considers the following problem: Find the minimum $f(k)$ such that a $f(k)$-strongly connected tournament is k^*-strongly (or k^*-weakly) connected.

We prove that for $k \geq 1$, a $2k$-strong tournament which has at least one path of length 2 between any two vertices x and y is $(k+2)^*$-weakly connected, and that for $k \geq 0$, a $(2k+1)$-strong tournament is $(k+2)^*$-weakly connected. Further, for $k \geq 2$, we prove that a $2k$-strong tournament is k^*-strongly connected, and that a $(2k+1)$-strong tournament which has at least one 2-bypass for each arc is $(k+1)^*$-strongly connected. Finally, we also prove that in a tournament with n vertices and irregularity $i(T) \leq k$, if $n \geq 6t + 5k$ ($t \geq 2$), then $\kappa_s^*(T) \geq t$ and that in a tournament with n vertices and irregularity $i(T) \leq k$, if $n \geq 6t + 5k - 3$ ($t \geq 2$), then $\kappa_w^*(T) \geq t + 1$.

2. Preliminary

The following is easy to obtain.

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