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Algorithmic aspects of rotor-routing and the notion of linear equivalence

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ABSTRACT

We define the analogue of linear equivalence of graph divisors for the rotor-router model, and use it to prove polynomial time computability of some problems related to rotor-routing. Using the connection between linear equivalence for chip-firing and for rotor-routing, we give a simple proof for the fact that the number of rotor-router unicycle-orbits equals the order of the Picard group. We also show that the rotor-router action of the Picard group on the set of spanning in-arborescences can be interpreted in terms of the linear equivalence.

1. Introduction

Rotor-routing is a deterministic process that induces a walk of a chip on a directed graph. It was introduced in the physics literature as a model of self-organized criticality [12,13,4]. The rotor walk can also be thought of as a derandomized random walk on a graph [7].

In this paper, we explore the relationship of rotor-routing with the chip-firing game, and the Picard group of the graph. We analyze a generalized version of rotor-routing, where each vertex has an integer number of chips, which might also be negative. This model has sometimes been called the height-arrow model [3]. Rotor-routing in this setting becomes a one-player game analogous to chip-firing, where a vertex can make a step if it has a positive number of chips.

In Section 2, we characterize recurrent elements for the rotor-routing game. This result is a generalization of a result of Holroyd et al. [6] that characterizes recurrent configurations with one chip. A motivation for such a characterization is the fact that for the chip-firing game, no characterization is known for the recurrent elements on general digraphs.

In Section 3, we define the analogue of the notion of linear equivalence of the chip-firing game for the rotor-routing game. We show that the linear equivalence notions of the two models are related in a simple way. Moreover, whether two configurations of the rotor-routing game are linearly equivalent can be decided in polynomial time.

We use this result to prove polynomial time decidability of the reachability problem for rotor-routing in a special case. In particular, we show, that it can be decided in polynomial time whether two unicycles lie in the same rotor-router orbit. Using the relationship between linear equivalence for chip-firing and for rotor-routing, we give a simple bijective proof for the fact that the number of rotor-router unicycle orbits equals the order of the Picard group of the graph. (This fact also follows from a combination of previous results [11, Theorem 1] and [5, Theorem 2.10], but they do not provide a bijection.) Finally, we show, that the rotor-router action of the Picard group on the set of spanning in-arborescences [6] can also be interpreted in terms of the linear equivalence. Using this interpretation, we show that it can be checked in polynomial time, whether a given spanning in-arborescence is the image of another given arborescence by a given element of the Picard group.

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Also using this interpretation, we give a simpler proof for the result of Chan et al. [2] stating that the rotor-router action is independent of the base point if and only if all cycles in the graph are reversible.

1.1. Basic notations

Throughout this paper, *digraph* means a directed graph, where multiple edges are allowed, but there are no loops. We will almost always assume our digraphs to be strongly connected. For a digraph G, $V(G)$ denotes the set of vertices, and $E(G)$ denotes the set of edges. For a directed edge $\overrightarrow{uv}$, u is the *tail*, and v is the *head*. The multiplicity of the edge $\overrightarrow{uv}$ is denoted by $d(u, v)$. We denote the set of out-neighbors (in-neighbors) of a vertex v by $\Gamma^+(v)$ ($\Gamma^-(v)$), the out-degree (in-degree) of a vertex v by $d^+(v)$ ($d^-(v)$).

For a digraph G and vertex $w \in V(G)$ a *spanning in-arborescence* of G rooted at w is a subdigraph G' such that $d^+_{G'}(v) = 1$ for each $v \in V(G) - w$, and the underlying undirected graph of G' is a tree.

We denote by $\mathbb{Z}^{V(G)}$ the set of integer vectors indexed by the vertices of a digraph G. We identify vectors in $\mathbb{Z}^{V(G)}$ with integer valued functions on $V(G)$. According to this, we write $z(v)$ for the coordinate corresponding to vertex v of a $z \in \mathbb{Z}^{V(G)}$. We denote by $z \geq 0$ if a vector $z \in \mathbb{Z}^{V(G)}$ is coordinatewise nonnegative. We use the notation $\mathbf{0}_G$ ($\mathbf{1}_G$) for the vector where each coordinate equals zero (one). We denote the characteristic vector of a vertex v by $\mathbf{1}_v$.

Definition 1.1. The *Laplacian matrix* of a digraph G is the following matrix $L_G \in \mathbb{Z}^{V(G)\times V(G)}$:

$$L_G(u, v) = \begin{cases} -d^+(u) & \text{if } u = v, \\ d(v, u) & \text{if } u \neq v. \end{cases}$$

Proposition 1.2 ([1, Proposition 4.1 and 3.1]). *For a strongly connected digraph G, there exists a unique vector* $\mathrm{per}_G \in \mathbb{Z}^{V(G)}$ *such that $L_G \mathrm{per}_G = \mathbf{0}_G$, the entries of* per_G *are strictly positive, and relatively prime. If G is Eulerian, then* $\mathrm{per}_G = \mathbf{1}_G$.

The vector per_G is called the primitive period vector of G.

1.2. Chip-firing

Chip-firing is a solitary game on a directed graph. The configurations of the game are called *divisors*. A divisor x is an integer vector indexed by the vertices of the graph, i.e. $x \in \mathbb{Z}^{V(G)}$. We think of $x(v)$ as the number of chips on vertex v (which might be negative). The *degree* of a divisor is the sum of its entries: $\deg(x) = \sum_{v\in V(G)} x(v)$. We denote the set of divisors on a digraph G by $\mathrm{Div}(G)$, and the set of divisors of degree k by $\mathrm{Div}^k(G)$. Note that $\mathrm{Div}(G)$ and $\mathrm{Div}^0(G)$ are Abelian groups with the coordinatewise addition.

The basic operation in the game is a *firing* of a vertex. For a divisor x, firing a vertex v means taking the new divisor $x' = x + L_G\mathbf{1}_v$, i.e, v loses $d^+(v)$ chips, and each out-neighbor u of v receives $d(v, u)$ chips. Note that a firing preserves the degree of the divisor.

The firing of a vertex v is *legal* with respect to the divisor x, if $x(v) \geq d^+(v)$, i.e, if the vertex v has a nonnegative number of chips after the firing. (Note that other vertices might have a negative number of chips.) A *legal game* is a sequence of divisors in which each divisor is obtained from the previous one by a legal firing.

The following equivalence relation on $\mathrm{Div}(G)$, called *linear equivalence*, plays an important role in the theory of chip-firing: $x \sim y$ if there exists an integer vector $z \in \mathbb{Z}^{V(G)}$ such that $y = x + L_G z$. One can easily check that this is indeed an equivalence relation. As per_G is a strictly positive eigenvector of L_G with eigenvalue zero, we can suppose that $z \geq 0$: We have $L_G(z + k \cdot \mathrm{per}_G) = L_G z$ for any $k \in \mathbb{Z}$, and for a sufficiently large k, $z + k \cdot \mathrm{per}_G \geq 0$. Thus $x \sim y$ if and only if y can be reached from x by a sequence of (not necessarily legal) firings.

Note that the divisors linearly equivalent to $\mathbf{0}_G$ form a subgroup of $\mathrm{Div}^0(G)$ which is isomorphic to $\mathrm{Im}(L_G)$, the image of the linear operator on $\mathbb{Z}^{V(G)}$ corresponding to L_G. The factor group of $\mathrm{Div}^0(G)$ by linear equivalence is called the *Picard-group* of the graph:

$$\mathrm{Pic}^0(G) = \mathrm{Div}^0(G)/\mathrm{Im}(L_G) .$$

1.3. Rotor-routing

The rotor-routing game is played on a ribbon digraph. A *ribbon digraph* is a digraph together with a fixed cyclic ordering of the outgoing edges from v for each vertex v. For an edge $e = \overrightarrow{vw}$, denote by e^+ the edge following e in the cyclic order at v. From this point, we always assume that our digraphs have a ribbon digraph structure.

Let G be a ribbon digraph. A *rotor configuration* on G is a function ϱ that assigns to each non-sink vertex v an out-edge with tail v. We call $\varrho(v)$ the *rotor* at v. For a rotor configuration ϱ, we call the subgraph with edge set $\{\varrho(v) : v \in V(G)\}$ the *rotor subgraph*.

A configuration of the rotor-routing game is a pair (x, ϱ), where $x \in \mathrm{Div}(G)$ is divisor, and ϱ is a rotor configuration on G. We also call such pairs *divisor-and-rotor configuration*, or just shortly DRC.