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Note

Terminal-pairability in complete bipartite graphs

Lucas Colucci [b], Péter L. Erdős [a], Ervin Győri [a,b], Tamás Róbert Mezei [a,b,*]

[a] Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda u. 13–15, 1053 Budapest, Hungary
[b] Central European University, Department of Mathematics and its Applications, Nádor u. 9, 1051 Budapest, Hungary

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ABSTRACT

We investigate the terminal-pairability problem in the case when the base graph is a complete bipartite graph, and the demand graph is also bipartite with the same color classes. We improve the lower bound on maximum value of $\Delta(D)$ which still guarantees that the demand graph D is terminal-pairable in this setting. We also prove a sharp theorem on the maximum number of edges such a demand graph can have.

1. Introduction

The *terminal-pairability* problem has been introduced in [1]. It asks the following question: given a simple *base graph* G and a list of pairs of vertices of G (which list may contain multiple copies of the same pair), can we assign to each pair a path in G whose end-vertices are the two elements of the pair, such that the set of chosen paths are pairwise edge-disjoint.

The above problem can be compactly described by a pair of graphs: the base graph and a so-called *demand graph*, which is a loopless multigraph on the same set of vertices as the base graph together with the list of pairs to be joined as the (multi)set of edges. If the terminal-pairability problem defined by D and G can be solved, then we say that D is resolvable in G. In this paper, demand graphs are denoted by D, or its primed and/or indexed variants.

Related to the terminal-pairability problem is the notion of weak linkedness, which is closely tied to the edge-connectivity number (see [6]). A graph G is weakly-k-linked if and only if every demand graph on $V(G)$ with at most k edges is resolvable G. In the terminal-pairability context, however, we are more interested in the degrees of D.

Given an edge $e \in E(D)$ with endvertices x and y, we define the *lifting of e to a vertex $z \in V(D)$*, as an operation which transforms D by deleting e and adding two new edges joining xz and zy; in case $z = x$ or $z = y$, the operation does not do anything. We stress that we do not use any information about G to perform a lifting and that the graph obtained using a lifting operation is still a demand graph.

Notice that the terminal-pairability problem defined by G and D is solvable if and only if there exists a series of liftings, which, applied successively to D, results in a (simple!) subgraph of G. This subgraph is called a *resolution* of D in G. The edge-disjoint paths can be recovered by assigning pairwise different labels to the edges of D, and performing the series of liftings so that new edges inherit the label of the edge they replace. Clearly, edges sharing the same label form a walk between the endpoints of the demand edge of the same label in D, and so there is also such a path.

This problem has been studied, for example, for complete graphs [1,4] and cartesian product of complete graphs [5,9]. In this paper we deal with problems where the base graph is a complete bipartite graph and the demand graph is bipartite with the color classes of the base graph.

* Corresponding author at: Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda u. 13–15, 1053 Budapest, Hungary.
E-mail addresses: colucci_lucas@phd.ceu.edu (L. Colucci), erdos.peter@renyi.mta.hu (P.L. Erdős), gyori.ervin@renyi.mta.hu (E. Győri), mezei.tamas.robert@renyi.mta.hu (T.R. Mezei).

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Conjecture 1 ([3]). *Let D be a bipartite demand graph whose base graph is $K_{n,n}$, i.e., $V(D) = V(K_{n,n})$ and each element of $E(D)$ is a copy of an edge of $K_{n,n}$. If $\Delta(D) \leq \lceil n/3 \rceil$ holds, then D is resolvable in $K_{n,n}$.*

The above conjecture is sharp in the sense that the disjoint union of n pairs of vertices each joined by $\lceil n/3 \rceil + 1$ parallel edges cannot be resolved in $K_{n,n}$, as explained by the following reasoning. From each set of edges joining the same pair of vertices at most one edge is resolved into a path of length 1 (itself), while the rest of them must be replaced by paths of length at least 3, therefore any resolution uses at least $n + 3 \cdot n \cdot \lceil n/3 \rceil \geq n^2 + n$ edges in $K_{n,n}$, which is a contradiction.

By replacing $\lceil n/3 \rceil$ with $n/12$ in Conjecture 1, we get a theorem of Gyárfás and Schelp [3]. We also cannot prove Conjecture 1 in its generality, but in the following theorem we improve the previous best known bound of $n/12$ to $(1 - o(1))n/4$.

Theorem 2. *Let D be a bipartite demand graph whose two color classes A and B have sizes a and b, respectively. If $d(x) \leq (1 - o(1))b/4$ for all $x \in A$ and $d(y) \leq (1 - o(1))a/4$ for all $y \in B$, then D is resolvable in the complete bipartite graph with color classes A and B.*

For certain graph classes, if n is divisible by 3, we can prove that the sharp bound $n/3$ holds. Let $\uplus$ denote the disjoint union of sets.

Theorem 3. *Let D be a bipartite demand graph with base graph $K_{n,n}$, such that*

$$U = \biguplus_{i=1}^{3} U_i \text{ and } V = \biguplus_{i=1}^{3} V_i$$

are the two color classes of D with $|U_i| = |V_i| \geq \lfloor \frac{n}{3} \rfloor$ for $i = 1, 2, 3$. If $\Delta(D) \leq \lfloor \frac{n}{3} \rfloor$ and for any $i \neq j$ there is no edge of D joining some vertex of U_i to some vertex of V_j, then D is resolvable in $K_{n,n}$.

Additionally, we prove a sharp bound on the maximum number of edges in a resolvable bipartite demand graph:

Theorem 4. *Let $n \geq 4$ and D be a bipartite demand graph with the base graph $K_{n,n}$. If D has at most $2n - 2$ edges and $\Delta(D) \leq n$, then D is resolvable in $K_{n,n}$.*

Notice the assumption $\Delta(D) \leq n$ is necessary: there can be at most n edge-disjoint paths starting at any given vertex. The result is sharp, as it is shown by the demand graph composed of a pair of vertices joined by n edges, another pair of vertices joined by $n - 1$ edges, and $2n - 4$ isolated vertices: in any resolution, one of the paths corresponding to one of the n edges joining the first pair of vertices passes through a vertex of the pair of vertices joined by $n - 1$ edges, implying that this vertex has degree $\geq n + 1$ in the resolution, a contradiction.

2. Proofs of the degree versions (Theorems 2 and 3)

Theorem 3 serves a dual purpose in our analysis: it provides several examples where Conjecture 1 holds and it demonstrates the techniques that will be used in the proof of Theorem 2. Before we proceed to prove the theorems, we state several definitions and three well-known results about edge-colorings of multigraphs.

Let H be a loopless multigraph. Recall that the chromatic index (or the edge chromatic number) $\chi'(H)$ is the minimum number of colors required to properly color the edges of a graph H. Similarly, the list chromatic index (or the list edge chromatic number) $\text{ch}'(H)$ is the smallest integer k such that if for each edge of G there is a list of k different colors given, then there exists a proper coloring of the edges of H where each edge gets its color from its list. The maximum multiplicity $\mu(H)$ is the maximum number of edges joining the same pair of vertices in H. The number of edges joining a vertex $x \in V(H)$ to a subset $A \subseteq V(H)$ of vertices is denoted by $e_H(x, A)$. The set of neighbors of x in H is denoted by $N_H(x)$. For other notation the reader is referred to [2].

Theorem 5 (Kőnig [8]). *For any bipartite multigraph H we have $\chi'(H) = \Delta(H)$, or, in other words, the edge set of H can be decomposed into $\Delta(H)$ matchings.*

Theorem 6 (Vizing, [10]). *For any multigraph H*

$$\chi'(H) \leq \Delta(H) + \mu(H).$$

Theorem 7 (Kahn, [7]). *For any multigraph H*

$$\text{ch}'(H) \leq (1 + o(1))\chi'(H).$$

Even though in our theorems the demand graphs are bipartite, in the proofs we may transform them into non-bipartite ones.

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