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On the bend number of circular-arc graphs as edge intersection graphs of paths on a grid

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Dedicated to Martin Charles Golumbic on the occasion of his 65th birthday

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ABSTRACT

Golumbic, Lipshteyn and Stern [12] proved that every graph can be represented as the edge intersection graph of paths on a grid (EPG graph), i.e., one can associate with each vertex of the graph a nontrivial path on a rectangular grid such that two vertices are adjacent if and only if the corresponding paths share at least one edge of the grid. For a nonnegative integer k, B_k-EPG graphs are defined as EPG graphs admitting a model in which each path has at most k bends. Circular-arc graphs are intersection graphs of open arcs of a circle. It is easy to see that every circular-arc graph is a B_4-EPG graph, by embedding the circle into a rectangle of the grid. In this paper, we prove that circular-arc graphs are B_3-EPG, and that there exist circular-arc graphs which are not B_2-EPG. If we restrict ourselves to rectangular representations (i.e., the union of the paths used in the model is contained in the boundary of a rectangle of the grid), we obtain EPR (edge intersection of paths in a rectangle) representations. We may define B_k-EPR graphs, $k \geq 0$, the same way as B_k-EPG graphs. Circular-arc graphs are clearly B_4-EPR graphs and we will show that there exist circular-arc graphs that are not B_3-EPR graphs. We also show that normal circular-arc graphs are B_2-EPR graphs and that there exist normal circular-arc graphs that are not B_1-EPR graphs. Finally, we characterize B_1-EPR graphs by a family of minimal forbidden induced subgraphs, and show that they form a subclass of normal Helly circular-arc graphs.

1. Introduction

Let $\mathcal{G}$ be a rectangular grid of size $(\ell + 1) \times (\ell + 1)$. The vertical grid lines will be referred to as *columns* and denoted by $x_0, x_1, \ldots, x_\ell$, and the horizontal grid lines will be referred to as *rows* and denoted by $y_0, y_1, \ldots, y_\ell$. A grid point lying on column x and row y is referred to as (x, y). A path on $\mathcal{G}$ is *nontrivial* if it contains at least one edge of the grid. Let $\mathcal{P}$ be a

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collection of nontrivial simple paths on $\mathcal{G}$. The edge intersection graph of $\mathcal{P}$ (denoted by EPG($\mathcal{P}$)) is the graph whose vertices correspond to the paths of $\mathcal{P}$ and two vertices are adjacent in EPG($\mathcal{P}$) if and only if the corresponding paths in $\mathcal{P}$ share at least one edge in $\mathcal{G}$. A graph G is called an *edge intersection graph of paths on a grid (EPG graph)* if $G = \text{EPG}(\mathcal{P})$ for some $\mathcal{P}$. Every graph G satisfies $G = \text{EPG}(\mathcal{P})$ for some $\mathcal{P}$ on a large enough grid and allowing an arbitrary number of *bends* (turns on a grid point) for each path [12]. In recent years, the subclasses for which the number of bends of each path is bounded by some integer $k \geq 0$, known as *B_k-EPG graphs*, were widely studied [2–4,8,12,14,15]. The *bend number* of a graph G (resp. a graph class $\mathcal{H}$), is the smallest integer $k \geq 0$ such that G (resp. every graph in $\mathcal{H}$) is a B_k-EPG graph. We denote by B_k-EPG, $k \geq 0$, the class of B_k-EPG graphs.

In [14], it was shown that for every integer $k \geq 0$ there exists a graph with bend number k, and that recognizing B_1-EPG graphs is NP-complete. The bend number of classical graph classes was investigated as well. In [15], it was shown that outerplanar graphs are B_2-EPG graphs and that planar graphs are B_4-EPG graphs. For planar graphs, it is still an open question whether their bend number is equal to 3 or 4. On the other hand, it is easy to see that B_0-EPG graphs exactly correspond to interval graphs (i.e., intersection graphs of intervals on a line) [12]. A generalization of interval graphs is circular-arc (CA) graphs, i.e., intersection graphs of open arcs on a circle. It is natural to see circular-arc graphs as EPG graphs by identifying the circle with a rectangle of the grid. Hence, circular-arc graphs form a subclass of B_4-EPG graphs. This leads to some natural questions. For example, the bend number of circular-arc graphs or the characterization of circular-arc graphs that are B_k-EPG graphs, for some $k < 4$. One of the main results of this paper is that the bend number of circular-arc graphs is 3.

Another interesting question is how many bends per path are needed for a circular-arc graph to be represented in a rectangle of the grid, i.e., in such a way that the union of the paths is contained in the boundary of a rectangle of the grid. We call such graphs *edge intersection graphs of paths on a rectangle* (*EPR graphs*). It is easy to see that EPR graphs are exactly the circular-arc graphs. We will study the classes B_k-EPR, for $0 \leq k \leq 4$, in which the paths on the grid that represent the vertices of the graph have at most k bends. As before, we denote by B_k-EPR, $k \geq 0$, the class of B_k-EPR graphs. Similar to the case of EPG graphs, one can define for a circular-arc graph G the bend number with respect to an EPR representation as the smallest integer k such that G is a B_k-EPR graph. Notice that CA = EPR = B_4-EPR. We strengthen this observation by showing that the bend number for circular-arc graphs with respect to EPR representations is 4. Furthermore, we focus on B_1-EPR graphs and B_2-EPR graphs (B_0-EPR graphs correspond again to interval graphs), and relate these classes with the class of normal Helly circular-arc graphs. In summary, we obtain the following results: we prove that the bend number of normal circular-arc graphs with respect to EPR representations is 2; moreover, we characterize B_1-EPR graphs by a family of minimal forbidden induced subgraphs, and show that they are exactly the normal Helly circular-arc graphs containing no powers of cycles C_{4k-1}^k, with $k \geq 2$, as induced subgraphs.

An extended abstract of a preliminary version of this work was published in the proceedings of LAGOS 2015 [1].

2. Preliminaries

All graphs that we consider in this paper are connected, finite and simple. For all graph theoretical terms and notations not defined here, we refer the reader to [5].

We denote by C_n, $n \geq 3$, the chordless cycle on n vertices. A graph is called *chordal*, if every cycle of length at least four has a chord. Given a graph G and an integer $k \geq 0$, the *power graph* G^k has the same vertex set as G, two vertices being adjacent in G^k if their distance in G is at most k.

Let $G = (V, E)$ be a graph and let $X \subseteq V$. We denote by $G - X$ the subgraph of G induced by the vertex set $V - X$.

A *clique* (resp. a *stable set*) is a subset of vertices that are pairwise adjacent (resp. non adjacent). We say that a vertex v *dominates* a vertex w if they are adjacent and every neighbor of w is also a neighbor of v.

A *thick spider* S_n, $n \geq 2$, is the graph whose $2n$ vertices can be partitioned into a clique $K = \{c_1, \ldots, c_n\}$ and a stable set $S = \{s_1, \ldots, s_n\}$ such that, for $1 \leq i, j, \leq n$, c_i is adjacent to s_j if and only if $i \neq j$. Notice that S_{n_1} is an induced subgraph of S_{n_2} if $n_1 \leq n_2$. (The name *spider* for graphs or graph classes has been used in the literature with different meanings. We follow the notation in [16], in the particular case in which the *head* of the spider is empty.)

Given a circle $\mathcal{C}$ of length ℓ, we can assign to vertices $s_1, \ldots, s_n$ of the thick spider S_n a set of pairwise disjoint arcs of $\mathcal{C}$, each of them of length $\ell/n - 2\varepsilon$, and to vertices $c_1, \ldots, c_n$ of S_n a set of arcs of $\mathcal{C}$ of length $(n-1)\ell/n + \varepsilon$ each (where ε is a small enough real number), in such a way that the arc corresponding to c_i is disjoint from the arc corresponding to s_i and intersects every other arc corresponding to a vertex in S, for $i = 1, \ldots, n$. Notice that since the length of each of the arcs corresponding to vertices in K is greater than $\ell/2$, they are pairwise intersecting. So, S_n is a circular-arc graph, as we have described a *circular-arc model* for it.

More in general, if G is a circular-arc graph, $\mathcal{C}$ denotes the corresponding circle, and $\mathcal{A}$ the corresponding set of open arcs, then $(\mathcal{A}, \mathcal{C})$ is called a *circular-arc model* of G [20]. A graph G is a *Helly circular-arc graph* (*HCA graph*) [10] if it is a circular-arc graph having a circular-arc model such that any subset of pairwise intersecting arcs has a common point on the circle. Such a model is called a *Helly model*. A circular-arc graph having a circular-arc model without two arcs covering the whole circle is called a *normal circular-arc graph* (*NCA graph*), and such a model is called a *normal model*. Circular-arc models that are at the same time normal and Helly are precisely those without three or less arcs covering the whole circle (see, for example, Theorem 1 in [18]). A graph that admits such a model is called a *normal Helly circular-arc graph* (*NHCA graph*) [17]. We will denote by NCA (resp. NHCA) the class of normal (resp. normal Helly) circular-arc graphs.

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