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On the dominating induced matching problem: Spectral results and sharp bounds

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ABSTRACT

A matching M is a dominating induced matching of a graph if every edge is either in M or has a common end-vertex with exactly one edge in M. The extremal graphs on the number of edges with dominating induced matchings are characterized by its Laplacian spectrum and its principal Laplacian eigenvector. Adjacency, Laplacian and signless Laplacian spectral bounds on the cardinality of dominating induced matchings are obtained for arbitrary graphs. Moreover, it is shown that some of these bounds are sharp and examples of graphs attaining the corresponding bounds are given.

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1. Introduction

Throughout this paper we consider undirected simple graphs G of order $n > 1$ with a vertex set $V(G)$ and edge set $E(G)$. An element of $E(G)$, which has the vertices i and j as end-vertices, is denoted by ij. A matching M of G is a *dominating induced matching* (say a DIM) of G if every edge of G is either in M or has a common end-vertex with exactly one edge in M. A DIM is also called an *efficient edge domination set* (see for instance [10]). Observe that if M is a DIM of G, then there is a partition of $V(G)$ into two disjoint subsets $V(M)$ and S, where S is an independent set. Conversely, if there exists a graph G such that its vertex set $V(G)$ can be partitioned into two vertex subsets V_1 and V_2, where V_1 induces a matching and V_2 is an independent set, then the subset $M \subset E(G)$ of edges with both ends in V_1 is a DIM. Not all graphs have a DIM, for instance the cycle with four vertices C_4 has no DIM. The *DIM problem* asks whether a given graph has a dominating induced matching.

Dominating induced matchings have been studied, not always under the same designation, in [2,4,5,8,7,14,13,15]. The DIM problem is related with several practical applications. Some of them, as parallel resource allocation of parallel processing systems, encoding theory and network routing, as well as its relation with the 3-colorability problem are referred in [12]. In [12], it is also highlighted that graphs with dominating induced matchings are particular *polar graphs*. Notice that a polar graph is a graph where its vertex set can be partitioned into vertex subsets such that some are disjoint cliques and the others are independent sets with complete links between them [17]. Regarding its theoretical complexity, the DIM problem is NP-complete [10]. However, in [12] it is conjectured that unless $P = NP$, the DIM problem is polynomial-time solvable in the class of M-free graphs (where M is a finite set of graphs) if and only if M contains a graph from the class of graphs such that every connected component corresponds to a long claw, that is, a connected graph with a central vertex of degree three,

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three vertices of degree one, and all the remaining vertices have degree two (that is, formed by three paths starting from a central vertex). In fact, the sufficient condition was proved in [7], but the necessary one remains open.

This paper is devoted to the study of the DIM problem from the graph spectra point of view. Next, for the reader convenience, we introduce some of the basic concepts and notation used throughout the paper. For the remaining terminology from graph theory, including spectral graph theory, the reader is referred to the book [9].

The adjacency matrix of a graph G of order n is the $n \times n$ symmetric matrix $A(G) = (a_{ij})$ where $a_{ij} = 1$ if $ij \in E(G)$ and $a_{ij} = 0$ otherwise, respectively. The Laplacian (signless Laplacian) matrix of G is the matrix $L(G) = D(G) - A(G)$ ($Q(G) = D(G) + A(G)$), where $D(G)$ is the $n \times n$ diagonal matrix of vertex degrees of G. The matrices $A(G)$, $L(G)$ and $Q(G)$ are all real and symmetric. From Geršgorin's theorem, it follows that the eigenvalues of $L(G)$ and $Q(G)$ are nonnegative real numbers. The spectrum of $A(G)$, $L(G)$ and $Q(G)$ is denoted by $\sigma_A(G)$, $\sigma_L(G)$ and $\sigma_Q(G)$, respectively. In this text, $\sigma_A(G) = \{\lambda_1^{[i_1]}, \ldots, \lambda_p^{[i_p]}\}$, $\sigma_L(G) = \{\mu_1^{[j_1]}, \ldots, \mu_q^{[j_q]}\}$ and $\sigma_Q(G) = \{q_1^{[k_1]}, \ldots, q_r^{[k_r]}\}$ mean that λ_s, μ_s and q_s are an adjacency, Laplacian and signless Laplacian eigenvalue with multiplicity i_s, j_s or k_s. As usually, we denote the eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ in nonincreasing order, that is, $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$, $\mu_1(G) \geq \cdots \geq \mu_n(G)$ and $q_1(G) \geq \cdots \geq q_n(G)$. Considering a graph G, the largest eigenvalue of $A(G)$, $L(G)$ and $Q(G)$ will be denoted, respectively, by $\rho(A(G))$, $\rho(L(G))$ and $\rho(Q(G))$. As usually, $\rho(A(G))$ is called the index of G and it is also denoted $\rho(G)$. The associated eigenvectors are called the principal eigenvectors of $A(G)$, $L(G)$ or $Q(G)$, respectively. For an arbitrary square matrix C the ith eigenvalue and its trace are denoted by $\lambda_i(C)$ and $\mathrm{tr}(C)$, respectively. Throughout this paper, $\mathbf{j}_k$ denotes the all one vector with k entries and $t + \sigma(C)$ means that we add t to each eigenvalue in $\sigma(C)$.

Consider a graph G of order n with a DIM $M \subset E(G)$ such that $|M| = m$, where (as above) $V(G) = V_1 \cup V_2$, with $V_1 = V(M)$ and $V_2 = V(G) \setminus V_1$ is an independent set. The property of having a DIM does not change whether we add edges linking the vertices of V_1 with the vertices of V_2. The extremal graph G', obtained from G adding $m(2(n-2m)+1) - |E(G)|$ edges (which is the maximum as possible) between V_1 and V_2, that is, such that $E(G') = M \cup \{xy : x \in V(M), y \in V(G) \setminus V(M)\}$ is herein called a *complete dominating induced matching*, say a CDIM. These graphs are particular cases of cographs [3].

The paper is organized as follows. In Section 2, the extremal graphs CDIM, are characterized by its Laplacian spectrum and by its principal Laplacian eigenvector. Notice that this characterization is important since in general, as it is well know, co-spectral graphs (relatively to adjacency, Laplacian or signless Laplacian matrices) are not necessarily isomorphic. The principal adjacency and signless Laplacian eigenvectors are deduced. Additionally, the adjacency and signless Laplacian spectra of graphs with a CDIM are presented. In Section 3, adjacency, Laplacian and signless Laplacian spectral bounds on the cardinality of dominating induced matchings are obtained for arbitrary graphs. Moreover, it is shown that some of these bounds are sharp and examples of graphs attaining the corresponding bounds are given.

2. Adjacency, Laplacian and signless Laplacian spectra of graphs with a CDIM

Given a graph H of order n with a CDIM, M such that $|M| = m$, we may define H using the join graph operation as follows. Let $H_r = mK_2$, with $r = 2m$ and $H_s = G[V(G) \setminus V(M)]$, with $s = n - r$, a null graph of order s (that is, a graph formed by s isolated vertices). Then $H = H_r \vee H_s$, that is, H is the join of the graphs H_r and H_s.

Consider the two above vertex disjoint graphs H_r and H_s and label the vertices of $H = H_r \vee H_s$, with the labels $1, 2, \ldots, r$ for the vertices of H_r and with the labels $r+1, \ldots, r+s$, for the vertices of H_s. Let $C(H)$ be a matrix on $H = H_r \vee H_s$. If $C(H) = L(H)$ or $C(H) = A(H)$ or $C(H) = Q(H)$ then, using the above mentioned labeling for the vertices of H, we obtain

$$C(H) = \begin{bmatrix} C_1 & \delta \mathbf{j}_r \mathbf{j}_s^T \\ \delta \mathbf{j}_s \mathbf{j}_r^T & C_2 \end{bmatrix}, \quad (1)$$

where δ is a scalar parameter, $C_1 = A(H_r)$ and $C_2 = A(H_s)$ or $C_1 = L(H_r) + sI_r$ and $C_2 = L(H_s) + rI_s$ or $C_1 = Q(H_r) + sI_r$ and $C_2 = Q(H_s) + rI_s$, when $C(H)$ is the adjacency, Laplacian or signless Laplacian matrix of H, respectively. In any case, in (1) we have $\delta \in \{1, -1\}$. Notice that

$$C_1 \mathbf{j}_r = \gamma_1 \mathbf{j}_r \quad \text{and} \quad C_2 \mathbf{j}_s = \gamma_2 \mathbf{j}_s,$$

with $\gamma_1 = 1$ and $\gamma_2 = 0$ (when $C(H)$ is the adjacency matrix) or $\gamma_1 = s$ and $\gamma_2 = r$ (when $C(H)$ is the Laplacian matrix) or $\gamma_1 = 2 + s$ and $\gamma_2 = r$ (when $C(H)$ is the signless Laplacian matrix).

Let us consider the matrix

$$B = \begin{bmatrix} \gamma_1 & \delta\sqrt{rs} \\ \delta\sqrt{rs} & \gamma_2 \end{bmatrix}, \quad (2)$$

where $\delta = \pm 1$, and its eigenvalues

$$\theta_1 = \frac{1}{2}\left(\gamma_1 + \gamma_2 + \sqrt{(\gamma_1 - \gamma_2)^2 + 4rs}\right) \quad (3)$$

$$\theta_2 = \frac{1}{2}\left(\gamma_1 + \gamma_2 - \sqrt{(\gamma_1 - \gamma_2)^2 + 4rs}\right). \quad (4)$$

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